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Abstract

In this work we explore some ways of studying financial and social networks, a topic that has recently received tremendous amount of attention in the Econometric literature.

Chapter 2 studies risk spillover effect via Multivariate Conditional Autoregressive Value at Risk model introduced in White et al. (2015). We are particularly interested in application to non-stationary time series and develop a sequential test procedure that chooses the largest available interval of homogeneity. This allows to balance between bias that appears due to parameter shifts, when the estimation sample is too large, and the variance. Our approach is based on change point test statistics and we use a novel Multiplier Bootstrap approach for the evaluation of critical values. The properties of the estimator are successfully studied theoretically and through simulations. Applying the method to certain market indices we study the risk dependencies between the financial markets.

In Chapter 3 we aim at social networks. We model interactions between users through a vector autoregressive model, following Zhu et al. (2017). To cope with high dimensionality we consider a network that is driven by influencers on one side, and communities on the other, which helps us to estimate the autoregressive operator even when the number of active parameters is smaller than the sample size. The estimation procedure is based on combination of a greedy clustering algorithm and Lasso. With application to daily sentiment weights extracted from a microblogging platform StockTwits we are able to identify the important users.

Chapter 4 is devoted to technical tools related to covariance cross-covariance estimation. We derive uniform versions of the Hanson-Wright inequality for a random vector with independent subgaussian components. The core technique is based on the entropy method combined with truncations of both gradients of functions of interest and of the coordinates itself. The results recover, in particular, the classic uniform bound of Talagrand (1996) for Rademacher chaoses and a more recent uniform result of Adamczak (2015), which holds under certain rather strong assumptions on the distribution. We provide several applications of our techniques: we establish a version of the standard Hanson-Wright inequality, which is tighter in some regimes. Extending our results we show a version of the dimension-free matrix Bernstein inequality that holds for random matrices with a subexponential spectral norm. We apply the derived inequality to the problem of covariance estimation with missing observations and prove an improved high probability version of the recent result of Lounici (2014).

Keywords: conditional quantile autoregression, local parametric approach, change point detection, multiplier bootstrap, social media, network autoregression, influencer, community, sentiment analysis, StockTwits, concentration inequalities, modified logarithmic Sobolev inequalities, uniform Hanson-Wright inequalities, matrix Bernstein inequality

Zusammenfassung

In dieser Arbeit untersuchen wir einige Möglichkeiten, financial und soziale Netzwerke zu analysieren, ein Thema, das in letzter Zeit in der ökonometrischen Literatur große Beachtung gefunden hat.

Kapitel 2 untersucht den Risiko-Spillover-Effekt über das in White et al. (2015) eingeführte multivariate bedingtes autoregressives Value-at-Risk-Modell. Wir sind besonders an der Anwendung auf nicht stationäre Zeitreihen interessiert und entwickeln einen sequentiellen statistischen Test, welcher das größte verfügbare Homogenitätsintervall auswählt. Dies ermöglicht einen Kompromiss zwischen einer Verzerrung, die aufgrund von der Parameteränderung, wenn die Stichprobegröße zu großist auftritt, und der Varianz. Unser Ansatz basiert auf der Changepoint-Teststatistik und wir verwenden einen neuartigen Multiplier Bootstrap Ansatz zur Bewertung der kritischen Werte. Die Eigenschaften des Schätzers wurden theoretisch und durch Simulationen erfolgreich untersucht. Unter Anwendung der Methode auf bestimmte Marktindizes untersuchen wir die Risikoabhängigkeiten zwischen den Finanzmärkten.

In Kapitel 3 konzentrieren wir uns auf soziale Netzwerke. Wir modellieren Interaktionen zwischen Benutzern durch ein Vektor-Autoregressivmodell, das Zhu et al. (2017) folgt. Um für die hohe Dimensionalität kontrollieren, betrachten wir ein Netzwerk, das einerseits von Influencers und Andererseits von Communities gesteuert wird, was uns hilft, den autoregressiven Operator selbst dann abzuschätzen, wenn die Anzahl der aktiven Parameter kleiner als die Stichprobengr̈ße ist. Das Schätzverfahren basiert auf der Kombination eines Greedy-Clustering-Algorithmus und Lasso. Mit der Anwendung auf die täglichen Sentiment Gewichte, die von einer Microblogging-Plattform StockTwits extrahiert wurden, sind wir in der Lage, die wichtigen Benutzer zu identifizieren.

Kapitel 4 befasst sich mit technischen Tools für die Schätzung des Kovarianzmatrix und Kreuzkovarianzmatrix. Wir entwickeln eine neue Version von der Hanson-Wright-Ungleichung für einen Zufallsvektor mit subgaußschen Komponenten. Die Kerntechnik basiert auf der Entropiemethode in Kombination mit Kürzungen sowohl der Gradienten der interessierenden Funktionen als auch der Koordinaten selbst. Die Ergebnisse stützen sich insbesondere auf die klassische Uniformgrenze von Talagrand (1996) für Rademacher-Chaosen und ein neues Uniformergebnis von Adamczak (2015) das unter bestimmten ziemlich starken Voraussetzungen für die Verteilung gilt. Wir bieten verschiedene Anwendungen unserer Techniken an: Wir stellen eine Version der Standard-Hanson-Wright-Ungleichung auf, die in einigen Regimen besser ist. Ausgehend von unseren Ergebnissen zeigen wir eine Version der dimensionslosen Bernstein-Ungleichung, die für Zufallsmatrizen mit einer subexponentiellen Spektralnorm gilt. Wir wenden diese Ungleichung auf das Problem der Schätzung der Kovarianzmatrix mit fehlenden Beobachtungen an und beweisen eine verbesserte Version des früheren Ergebnisses von (Lounici 2014).

Schlagwörter: bedingtes autoregressives Value-at-Risk-Modell, lokaler parametrischer Ansatz, Changepoint-Test, Multiplier Bootstrap, social media, Netzwerk Autoregressivmodell, Influencer, Community, Sentiment Analysis, StockTwits, Konzetrationsungleichingen, modified-logarithmic-Sobolev-Ungleichungen, Uniform-Hanson-Wright-Ungleichungen, Matrix-Bernstein-Ungleichung

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Chapter 1

Introduction

Risk dependence within financial networks and the mechanism of risk spillover among international equity markets has attracted increasing attention among theorists, empirical researchers and practitioners. A risk contagion is generated through dependence between extreme negative shocks across financial markets. It is well-known that large downside market movements occurring in one country would unavoidably have substantial effects on other international equity markets. Moreover, financial risk scenarios tend to transmit themselves among different markets, which consequently intensifies a global risk contagion leading to an international economic crisis. Identifying sensitivity of financial institutions to shocks to the whole system is a vital task in controlling stability of financial markets. For this purpose White et al. (2015) introduces Multivariate Conditional Autoregressive Value at Risk (MV-CAViaR) model, which is typically applied pairwise between institutions and financial market indices. However, empirical studies suggest that interdependence of the tail risk contagion is unstable and time-varying, (Baele and Inghelbrecht, 2010; Elyasiani et al., 2007). The model, therefore, asks for a procedure that would balance between long-term biasness and short-term high variance of the estimator. In Chapter 2 we introduce and develop such procedure. Based on the idea of sequential testing from Spokoiny (2009), we pick a time interval that passes homogeneity test with a predefined confidence level. The homogeneity test is based on a multiscale change point test statistics. The latter requires simulation of critical values, since pivotal distribution is typically not given, plus we want as well to account for possible misspecification of a model. A novel approach based on Multiplier Bootstrap is used, Spokoiny and Zhilova (2015). We analyse the properties of this test both theoretically and through simulation study and apply it to a simultaneous CAViaR model of stock market indices DAX and S&P 500.

Social media is another type of networks that receives plenty of attention in the recent Econometric literature. It represents an ideal platform where users can easily communicate with each other, exchange information and share opinions. An increasing popularity in social media is a clear evidence of such demand for exchanging options and information among granular users in the cyber world. Econometric analysis of social media data encounters the challenges from the granularity of users, complexity of interaction and a variety of opinions. On the other hand, these challenges bear the chances to augment econometric analysis via the massive availability of social media data. In Chapter 3 we model interactions in a social network through a vector autoregressive model, following a line of work Zhu and Pan (2017); Zhu et al. (2017, 2016). Such a model naturally suffers from curse of dimensionality, as the number of connection within a typical network is often larger that the available data sample, due to either limited data or time-variation of the model parameter. To cope with this problem we take into account two major aspects of social networks. The first one relies on the fact that in a typical social network only a small portion of users produce significant influence on the network, whom we call influencers. Secondly, each user in a social network represents a large group of users called community, who together share opinions and exhibit similar behaviour. This motivates us to introduce a new model called Social Network with Influencers and Communities (SoNIC), bringing the two aspects together. In theoretical and simulation analysis we show that it allows consistent estimation even when the number of users is smaller than the available time period. We focus on the application to sentiment extracted from StockTwits, a microblogging platform dedicated to discussion of stock market assets for traders and financial analysts. Apart from the estimation of the network connections, we identify the influencers — important users whose opinion matters the most.

We additionally provide several theoretical extensions and improvements. In Chapter 2 we show a Bahadur-type expansion for quantile estimation with exponentially high probabilities in the finite sample regime. In the appendix in Section A.1 we extend the results of Tropp (2006) for the exact Lasso recovery in the case of missing observations. Finally, in Chapter 4 we prove a new version of Bernstein Matrix inequality that works for unbounded matrices. As an application we improve the tail bound of Lounici (2014) for the covariance estimator under missing observations. Using a similar trick we extend uniform Hanson-Wright inequality to general unbounded subgaussian variable, a problem closely related to covariance estimation.

Chapter 2

Localizing Multivariate Conditional Autoregressive Value at Risk

There exists a wide-spread consensus in the empirical literature that the dependence between the returns of financial assets is non Gaussian with asymmetric marginals, nonlinear features and time-varying (Longin and Solnik, 2001; Okimoto, 2008). In order to address these properties Engle and Manganelli (2004) propose a conditional autoregressive value at risk (CAViaR) model to specify the evolution of conditional quantile over time for univariate time series. Further, White et al. (2015) built up a multivariate framework for multiple time series as well as various quantile levels, which can be considered as a vector autoregressive (VAR) extension to quantile models with the underlying value at risk processes not only autocorrelated but also cross-sectionally intertwined. When applying CAViaR to financial institutions, it presents valuable results in capturing the sensitivity of financial entities to institutional specific and market-wide shocks of the system. It does however not cope with time-variation. We therefore propose a feasible extension towards a local multivariate CAViaR to estimate and forecast the dynamics of financial risk dependence.

The majority of existing literature use volatility as the risk measure and investigate the volatility risk contagions (e.g. Bauwens et al. (2006); Engle (2002, 2004); Pelletier (2006)). Although volatility is a crucial instrument to measure the risk movement, it has been commonly criticized as only capturing the properties of second moments of the return time series and ignoring extreme market events structure (Han et al., 2016; Hong et al., 2009). In addition, the volatility risk measure is symmetric and equally values the gains and losses, which contradicts the facts that investors tends to be more sensitive to the negative returns and especially for large downside risk, e.g. financial crisis. Therefore volatility risk measure is not enough to evaluate the financial risk interdependence. On the contrary, Value at Risk (VaR) is commonly utilized to measure the asymmetric risk due to the straightforward implications, i.e., evaluate the loss given a predetermined probability of extreme events. Although not a perfect risk measure, it has been accepted as a standard for financial regulations, e.g. a criterion by the Basel committee on banking supervision, Franke et al. (2019).

The interdependence of financial risk and especially the tail risk contagion is typically featured as unstable and time-varying by empirical studies (Baele and Inghelbrecht, 2010; Elyasiani et al., 2007). The risk contagion is caused by dependence between extreme negative shocks across international financial markets. A parametric model over a long-run time series is at limit to portray almost certainly existed properties of non-stationarity. Gerlach et al. (2011) propose a time-varying quantile model using a Bayesian approach for univariate time series. In this paper, we focus on time-varying parameter properties of multivariate quantile modelling. We propose a framework for localizing multivariate autoregressive conditional quantiles by exploiting a local parametric approach, denoted as LMCR model for simplicity. The advantages of our strategy are at least twofold: (1) we consider the extreme tail risk spillover among financial markets and (2) we examine interdependence pattern of the tail risk contagion, both in a dynamic time-varying context.

The local parametric approach (LPA) utilizes a parametric model over an adaptively chosen interval of homogeneity. The essential idea of LPA is to find — backwards looking — the longest interval that guarantees a relatively small modelling bias, see e.g. Spokoiny (1998, 2009). A great advantage of this modelling approach is the search of balance between the modelling bias and parameter variability, see e.g. Chen et al. (2010); Chen and Niu (2014); Härdle et al. (2015); Niu et al. (2017); Xu et al. (2018). Recent advances in multipliers bootstrap (MBS) allow to construct data-driven critical values for homogeneity tests based on change point detection, see Suvorikova and Spokoiny (2017) and the references therein. The MBS only relies on the autoregressive equation for conditional quantiles and has no particular assumption about the distribution of the innovations. In our research, we extend LPA to quantile regression and develop LMCR. In Section 2.1 we extend the asymptotic results of White et al. (2015) to finite samples. In particular, we establish a Bahadur-type expansion based on uniform exponential inequality Lemma 2.1, which may as well be of independent interest. We then compare it with the multiplier bootstrap counterpart by utilizing the results of Chernozhukov et al. (2013).

Our approach appears particularly suitable to capture the shifting asymmetric dependence among different markets. It is worth to mention that many papers appeared in the literature investigate the co-movements of large changes by utilizing the copula-based methods, see e.g. Chen and Fan (2006a,b); Zhang et al. (2016). Rather than relying on a concrete specification of a copula, we emphasize local parametric modelling of risk dependence via a multivariate CAViaR model. Moreover, a simulation study under various parameter change scenarios demonstrates the success of our method to recover time-varying parameter characteristics. In addition, when applying to the tail risk analysis of US and German market index, we find that at the 1% quantile level the typical LPA interval lengths in daily time series include on average 140 days. At the higher, 5% quantile level, the selected interval lengths range roughly between 160-230 days. This is of importance given the current historical simulation risk measures based on 250 days. Therefore this findings might change todays regulatory risk measurement tools. The model also presents appealing merits in forecasting the tail risk spillover when comparing with other competing for alternative approaches.

In what follows we first present the model and theoretical justification of parametric homogeneity test in Section 2.1. Section 2.3 introduce the local change point detection method. In Section 2.4, a simulation study examines the performance of our approach. Section 2.5 presents an empirical application. Finally, Section 2.6 concludes this paper.

2.1 Model

We consider a multivariate time series – typically, the log returns if financial institutions – $\mathscr{Y} = \{\mathbf{Y}_t : t = 1, \dots, T\}$, with each $\mathbf{Y}_t$ being a $n \times 1$ column. Denote the natural filtration $\mathscr{F}_t = \sigma\{\mathbf{Y}_1, \dots, \mathbf{Y}_t\}$ and we wish to estimate the quantiles of Y_{it} conditioned on $\mathscr{F}_{t-1}$ at any given moment $t = 1, \dots, T$.

The LMCR model, like CAViaR, assumes that conditional quantiles $q^*_{it} = \inf\{y : \mathsf{P}(Y_{it} \leq y | \mathscr{F}_{t-1}) \geq \tau_i\}$ follow the autoregressive equation

$$q^*_{it} = \Psi_t^\top \beta_i + \sum_{k=1}^{q} \sum_{j=1}^{n} \gamma_{ijk} q^*_{jt-k}, \tag{2.1}$$

where $\mathscr{F}_{t-1}$–measurable $\Psi_t \in \mathbb{R}^d$ denote predictors available at time t, which typically include lagged values of times series $\mathbf{Y}_t$. We have a parametric model with a finite-dimensional parameter $\theta = \big((\beta_i)_{i=1}^n, (\gamma_{ijk})_{i,j,k=1}^{n,n,q}\big) \in \mathbb{R}^{nd+n^2q}$. The modelling quantile functions are defined recursively,

$$q_{it}(\theta,\mathscr{Y}) = \Psi_t^\top \beta_i + \sum_{k=1}^{q}\sum_{j=1}^{n} \gamma_{ijk} q_{jt-k}(\theta,\mathscr{Y}). \tag{2.2}$$

For any interval $\mathscr{I} = [a,b] \subset \{0,\dots,T\}$ we will write

$$(Y_{it},\Psi_t)_{t\in\mathscr{I}} \sim \mathrm{LMCR}(\theta),$$

if the equation (2.1) is fulfilled on this interval with parameter θ.

The parameter can be estimated via the quantile regression quasi-Maximum Likelihood Estimator (qMLE). For a given quantile level of interest $\tau \in (0,1)$ denote the check function $\rho_\tau(x) = x(\tau - \mathbf{I}[1 \le \tau])$ and set

$$\ell_t(\theta) = -\sum_{i=1}^{n} \rho_\tau\{Y_{it} - q_{it}(\theta,\mathscr{Y})\},$$

— quasi log-probability of t's observation. The log-likelihood based on the interval $\mathscr{I} \subset \{1,\dots,T\}$ of observations for a fixed τ reads as

$$L_{\mathscr{I}}(\theta) = \sum_{t\in\mathscr{I}} \ell_t(\theta)$$

and the estimator based on this set of observations as

$$\widetilde{\theta}_{\mathscr{I}} = \arg\max_{\theta\in\Theta_0} L_{\mathscr{I}}(\theta). \tag{2.3}$$

The paper White et al. (2015) deals with the estimator that uses the whole data set $\mathscr{I} = \{1,\dots,T\}$ and provides consistency and asymptotic normality of the estimator when T tends to infinity.

Remark 2.1. *The value $-L_{\mathscr{I}}(\theta)$ is usually referred to as risk or contrast and the corresponding estimator as risk minimizer or contrast estimator. We, however, prefer the terms quasi likelihood and quasi maximum likelihood estimator, as we work with LRTs, Spokoiny and Zhilova (2015).*

The main objective of the present work is to provide a practical technique that chooses appropriate intervals $\mathscr{I}$. Roughly speaking, the longer the interval the less is the variance of the estimator, while choosing the interval too large we can bring in bias due to time-varying parameter. We say that the model is homogeneous at the time interval $\mathscr{I}$, if the following assumption holds.

Assumption 2.1. *There exists a "true" parameter $\theta^* \in \Theta_0$ such that $q_{it}^* = q_{it}(\theta^*, \mathscr{Y})$ for each $i = 1, \ldots, n$ and $t \in \mathscr{I}$.*

Obviously, such an assumption ensures that $\theta^* = \arg\max \mathsf{E}\ell_t(\theta)$ for each $t \in \mathscr{I}$, and, therefore, $\theta^* = \arg\max \mathsf{E}L_{\mathscr{I}}(\theta)$, which falls into the general framework of maximum likelihood estimators, see e.g. Huber (1967), White (1996) and Spokoiny (2017).

Here though we study LMCR, a non-stationary CAViaR model, that follows the *local parametric assumption*, meaning that for each time point t there exists a historical interval $[t-m;t]$ where the model is nearly homogeneous, we also derive the theoretical properties of LMCR under general mixing conditions which might be of interest by itself for a deeper stochastic analysis.

2.1.1 Assumptions

We first impose the following assumptions on the LMCR model, in particular, we say that the model is "homogeneous" on an interval $\mathscr{I}$ if it satisfies the assumptions of this section.

The first one ensures the identification of the model and is akin to Assumption 4 of White et al. (2015). The second one controls the values and derivatives of the quantile regression functions.

Assumption 2.2. *There is a set of indices $J \subset \{1, \ldots, n\}$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that whenever $\|\theta - \theta^*\| \geq \varepsilon$,*

$$\mathsf{P}\left(\cup_{i=1}^n \{|q_{it}(\theta) - q_{it}(\theta^*)| \geq \delta\}\right) \geq \delta, \qquad t \in \mathscr{I}. \tag{2.4}$$

Assumption 2.3. *(i) For $s = 0, 1, 2$ there are constants $D_s > 0$ such that for each i, t and for each $\theta \in \Theta_0$ it holds pointwise $|q_{it}(\theta, \cdot)| \leq D_0$, $\|\nabla q_{it}(\theta, \cdot)\| \leq D_1$ and $\|\nabla^2 q_{it}(\theta, \cdot)\| \leq D_2$. (ii) Conditional density of innovations ε_{it} are bounded from above $f_{it}(x) \leq f_0$ for each i, t and $x \in \mathbb{R}$. (iii) Additionally, conditional density of innovations satisfies $f_{it}(x) \geq \underline{f}$ for $|t| \leq \delta_0$.*

Furthermore, we impose the following assumptions on the given time series. Let us first recall the definition of the mixing coefficients. For any sub σ-fields $\mathscr{A}_1, \mathscr{A}_2$ of same probability space $(\Omega, \mathscr{F}, \mathsf{P})$ define,

$$\alpha(\mathscr{A}_1, \mathscr{A}_2) = \sup_{A \in \mathscr{A}_1, B \in \mathscr{A}_2} |\mathsf{P}(A \cap B) - \mathsf{P}(A)\mathsf{P}(B)|,$$
$$\beta(\mathscr{A}_1, \mathscr{A}_2) = \sup_{(A_i) \subset \mathscr{A}_1, (B_i) \subset \mathscr{A}_2} \sum_{i,j} \left|\mathsf{P}(A_i \cap B_j) - \mathsf{P}(A_i)\mathsf{P}(B_j)\right|,$$

where in the latter the supremum is taken over all finite partitions $(A_i) \subset \mathscr{A}_1$ and $(B_j) \subset \mathscr{A}_2$ of Ω. Then, the coefficients

$$a_k((X_t)) = \sup_t \alpha(\sigma(X_1, \ldots, X_t), \sigma(X_{t+k}, \ldots, X_T)),$$
$$b_k((X_t)) = \sup_t \beta(\sigma(X_1, \ldots, X_t), \sigma(X_{t+k}, \ldots, X_T))$$

and denote α– and β–mixing coefficients of the process $(X_t)_{t \le T}$, respectively.

Assumption 2.4. *(i) Suppose, that the sequence of vectors $(q_{\cdot t}(\theta), \nabla q_{\cdot t}(\theta))$ is α–mixing with $\alpha(m) \le \exp(-\gamma m)$ for some constant $\gamma > 0$; (ii) The sequence of vectors $\nabla q_{\cdot t}(\theta^*, \mathscr{Y})$ is β–mixing with coefficients $\beta(m) \le m^{-\delta}$, $\delta > 1$; (iii) for each $i = 1, \ldots, n$ the innovations ε_{it} for $t \in \mathscr{I}$ are i.i.d. and satisfy $\mathsf{P}(\varepsilon_{it} < 0) = \tau$.*

Finally, we introduce the assumptions concerning information matrix as well as variance of the score, which corresponds to Assumption 6 of White et al. (2015).

Assumption 2.5. *The vector $(\mathbf{q}_t^*, \nabla \mathbf{q}_t(\theta^*), \varepsilon_t)$ is a stationary process for $t \in \mathscr{I}$. Additionally, the matrices*

$$Q^2 = \mathsf{E} f_{it}(0) \nabla q_{it}(\theta^*)[\nabla q_{it}(\theta^*)]^\top, \qquad V^2 = \mathrm{Var}\{\mathbf{g}_t(\theta^*)\}$$

are strictly positive definite.

2.1.2 Consistency of the estimator

Here we present the results for consistency of the estimator $\widetilde{\theta}$ as the length of the interval $|\mathscr{I}|$ tends to infinity. Unlike White et al. (2015), who show convergence in probability or in square mean, we provide bounds with exponentially large probabilities, which allows us to take into consideration growing amount of intervals simultaneously.

One of the main tools in providing convergence and asymptotic normality for M-estimators is uniform deviation bounds for the score, see e.g. White (1996), Spokoiny

(2017) and the references therein. The *score* of the likelihood is $\nabla L_{\mathscr{I}}(\theta) = \sum_{t \in \mathscr{I}} \nabla \ell_t(\theta) = \sum_{t \in \mathscr{I}} \mathbf{g}_t(\theta)$, where we denote $\mathbf{g}_t(\theta) = \nabla \ell_t(\theta)$. By definition of the log-likelihood, we have $\mathbf{g}_t(\theta) = \sum_i \nabla q_{it}(\theta, \cdot)\psi_\tau\{Y_{it} - q_{it}(\theta, \cdot)\}$. We also introduce the expectation of the latter $\lambda_t(\theta) = \mathsf{E}\mathbf{g}_t(\theta)$. The following bound provides exponential in probability uniform deviation bound.

Lemma 2.1. *Assume 2.3 and 2.4 hold on an interval $\mathscr{I}$. Then,*

$$\sup_{\theta \in \Theta_0(\mathtt{r})} \frac{1}{|\mathscr{I}|^{1/2}} \left\| \sum_{t \in \mathscr{I}} \mathbf{g}_t(\theta) - \lambda_t(\theta) - \mathbf{g}_t(\theta^*) + \lambda_t(\theta^*) \right\| \le \diamondsuit(|\mathscr{I}|, \mathtt{r}, \mathtt{x}),$$

with probability at least $1 - e^{-\mathtt{x}}$, where

$$\diamondsuit(T', \mathtt{r}, \mathtt{x}) = C_1 \left\{ \mathtt{r}\sqrt{\mathtt{x}} + \mathtt{r}^{1/2}\sqrt{\mathtt{x} + \log T'} + T'^{-1/2}(\log T')^2(\mathtt{r}\mathtt{x} + \mathtt{x} + \log T') \right\}$$

with some C_1 that does not depend on $T', \mathtt{r}, \mathtt{x}$.

Remark 2.2. *Here the error term with $\mathtt{r}^{1/2}$ comes from the fact that $\mathbf{g}_t(\theta, \cdot)$ contains non-differentiable generalized errors $\psi_\tau(Y_{it} - q_{it}(\theta))$, which being Bernoulli random variables, can not be handled by chaining alone, unlike the case of smooth score, see e.g. Spokoiny et al. (2017).*

Given the result above we can bound the score uniformly over all parameter set. This allow us to have the following consistency result.

Proposition 2.1. *Let assumptions 2.1–2.5 hold on the interval $\mathscr{I}$. It holds with probability $\ge 1 - 6e^{-\mathtt{x}}$,*

$$\|\widetilde{\theta}_{\mathscr{I}} - \theta^*\| \le C_0 \sqrt{\frac{\mathtt{x} + \log|\mathscr{I}|}{|\mathscr{I}|}}.$$

2.1.3 Local quadratic expansion

The next step in providing asymptotic normality of the estimator $\widetilde{\theta}$ is a local Fisher expansion. The main tool is linear approximation of the gradient of the likelihood, which can be done by means of Proposition 2.1.

It is shown in White et al. (2015) (see formula (24)), that for each $\theta \in \Theta$,

$$\left\| \sum_{t \in \mathscr{I}} \lambda_t(\theta) - \sum_{t \in \mathscr{I}} \lambda_t(\theta^*) + |\mathscr{I}|Q^2(\theta - \theta^*) \right\| \le C_2 |\mathscr{I}| \|\theta - \theta^*\|^2, \tag{2.5}$$

with some C_2 that does not depend on the length of the interval. Finally, we present the main result of this section, that serves as a non-asymptotic adaptation of Theorem 2 of White et al. (2015). We postpone the proof to Section 2.7.3.

Proposition 2.2. *Suppose, on some interval $\mathscr{I} \subset [0,T]$ the Assumptions 2.1–2.5 hold. Then, for any $\mathrm{x} \le |\mathscr{I}|$, it holds with probability at least $1-3e^{-\mathrm{x}}$,*

$$\begin{aligned} \left\| \sqrt{|\mathscr{I}|} Q(\widetilde{\theta}_{\mathscr{I}} - \theta^*) - \xi_{\mathscr{I}} \right\| &\le C \frac{(\mathrm{x} + \log |\mathscr{I}|)^{3/4}}{|\mathscr{I}|^{1/4}}, \\ \left| L(\widetilde{\theta}_{\mathscr{I}}) - L(\theta^*) - \|\xi_{\mathscr{I}}\|^2/2 \right| &\le C \frac{(\mathrm{x} + \log |\mathscr{I}|)^{3/4}}{|\mathscr{I}|^{1/4}}, \end{aligned} \tag{2.6}$$

where $\xi_{\mathscr{I}} = \frac{1}{\sqrt{|\mathscr{I}|}} \sum_{t \in \mathscr{I}} Q^{-1} \mathbf{g}_t(\theta^*)$ and C does not depend on $|\mathscr{I}|$ and x.

Remark 2.3. *This result serves as a non-asymptotic version of central limit theorem (CLT) for the estimator, Theorem 2 in White et al. (2015). This follows from the fact that the sequence $(Q^{-1}\mathbf{g}_t(\theta^*))_{t \le T}$ satisfies CLT as a martingale difference sequence, see also Theorem 5.24 in White (2014).*

2.2 Homogeneity testing via local change point detection

Suppose, we have an interval $\mathscr{I} = [a,b] \subset \{1,\dots,T\}$ of observations and we want to test whether there is a change in the parameter, that generates the data on this interval through the model (2.1). An alternative would be that there exist a break point $s \in (a,b)$ such that on the left part $A_s = [a,s]$ the data generating process is described by one parameter and on the right part $B_s = [s+1,b]$ it is described by a different parameter. This means that we want to test a null hypothesis

$$\mathbf{H_0}(\mathscr{I}): \;\; (Y_{it}, \Psi_t)_{t \in \mathscr{I}} \sim \mathrm{LMCR}(\theta^*_{\mathscr{I}}),\; \theta^*_{\mathscr{I}} \in \Theta_0,$$

against the alternative

$$\begin{aligned} \mathbf{H_1}(\mathscr{I}): \;& (Y_{it}, \Psi_t)_{t \in \mathscr{I}} \sim \mathrm{LMCR}(\theta^*_{A_s}), \\ & (Y_{it}, \Psi_t)_{t \in \mathscr{I}} \sim \mathrm{LMCR}(\theta^*_{B_s}) \text{ with some } \theta^*_{A_s} \neq \theta^*_{B_s}. \end{aligned}$$

To construct the test statistics consider a set of candidates for a break point $\mathscr{S}(\mathscr{I}) \subset (a,b)$ and for each such candidate $s \in \mathscr{S}(\mathscr{I})$ introduce the test,

$$T_{\mathscr{I},s} = L_{A_{\mathscr{I},s}}(\widetilde{\theta}_{A_{\mathscr{I},s}}) + L_{B_{\mathscr{I},s}}(\widetilde{\theta}_{B_{\mathscr{I},s}}) - L_{\mathscr{I}}(\widetilde{\theta}_{\mathscr{I}}),$$

where $A_{\mathscr{I},s} = [a,s]$ represents observations to the left from break point and $B_{\mathscr{I},s} = [s+1,b]$ are the observations to the right from the break point candidate $s \in \mathscr{I}$. The existence of the break point among the candidates is tested using statistic

$$T_{\mathscr{I}} = \max_{s \in \mathscr{S}(\mathscr{I})} T_{\mathscr{I},s}.$$

Given a certain confidence level α we want to construct a critical value $\mathfrak{z}_{\mathscr{I},\alpha}$ such that under the null hypothesis it holds

$$\mathsf{P}\left(T_{\mathscr{I}} > \mathfrak{z}_{\mathscr{I},\alpha}\right) = \alpha,$$

which stands for the false alarm rate. Evaluating such critical values is a crucial question in hypothesis testing.

Spokoiny et al. (2013) and Xu et al. (2018) use a *propogation approach* for constructing the critical values. The approach is based on generation the distribution of test statistics, assuming that the distribution of the data is known precisely up to the parameter. For instance, the latter paper assumes normal distribution for the innovations in the conditional expectiles process. In the next section, in order to account for arbitrary distribution of the innovations, we construct data-driven critical values $\mathfrak{z}_{\mathscr{I},\alpha}(\mathscr{Y})$ that use the corresponding data interval for each test based on multiplier bootstrap.

2.2.1 Multiplier bootstrap

The idea is to simulate the unknown distribution of the original log-likelihood by introducing *MBS* with each term reweighted

$$L^{\circ}_{\mathscr{I}}(\theta) = \sum_{t \in \mathscr{I}} w_t \ell_t(\theta),$$

where $(w_t)_{t \leq T}$ is a given random sequence of i.i.d. weights independent of the sample. For sake of simplicity we additionally assume, that they have sub-Gaussian tails.

Assumption 2.6. *The weights w_t are independent with $\mathsf{E}w_t = 1$ and $\operatorname{Var}(w_t) = 1$. Additionally, there is C_w such that for each t it holds $\mathsf{E}\exp\{(w_t/C_w)^2\} \le 2$.*

Denote the corresponding bootstrap estimator

$$\widetilde{\theta}^{\circ}_{\mathscr{I}} = \operatorname{arg\,max} L^{\circ}_{\mathscr{I}}(\theta),$$

while the expectation of bootstrap log-likelihood with respect to the simulated weights is obviously maximized by the original estimator,

$$\widetilde{\theta}_{\mathscr{I}} = \operatorname{arg\,max} \mathsf{E}^{\circ} L^{\circ}_{\mathscr{I}}(\theta) = \operatorname{arg\,max} L_{\mathscr{I}}(\theta),$$

where $\mathsf{E}^{\circ}[\cdot] = \mathsf{E}[\cdot \,|\, \mathscr{Y}]$ denotes expectation in the "bootstrap world". The paper Spokoiny and Zhilova (2015) shows that with high probability the distribution of the simulated likelihood ratio $L^{\circ}_{\mathscr{I}}(\widetilde{\theta}^{\circ}_{\mathscr{I}}) - L^{\circ}_{\mathscr{I}}(\widetilde{\theta}_{\mathscr{I}})$ in the "bootstrap world" mimics the distribution of the original likelihood ratio $L_{\mathscr{I}}(\widetilde{\theta}_{\mathscr{I}}) - L_{\mathscr{I}}(\theta^*)$ up to some error that decreases with growing sample. We adapt their theory for the case of regression quantiles.

Proposition 2.3. *Suppose, Assumptions 2.1–2.5 and 2.6 hold on the interval $\mathscr{I}$. Then, there is $T_0 > 0$ such that if $T \ge T_0$ and $\mathtt{x} \le T$, on the event of probability at least $1 - e^{-\mathtt{x}}$, it holds with probability at least $1 - e^{-\mathtt{x}}$ conditioned on the data, that*

$$\begin{aligned}
\left\| \sqrt{|\mathscr{I}|} Q(\widetilde{\theta}^{\circ}_{\mathscr{I}} - \widetilde{\theta}_{\mathscr{I}}) - \xi^{\circ}_{\mathscr{I}} \right\| &\le C \frac{(\mathtt{x} + \log T)^{3/4}}{T^{1/4}}, \\
\left| L^{\circ}_{\mathscr{I}}(\widetilde{\theta}^{\circ}_{\mathscr{I}}) - L^{\circ}_{\mathscr{I}}(\widetilde{\theta}_{\mathscr{I}}) - \|\xi^{\circ}_{\mathscr{I}}\|^2/2 \right| &\le C \frac{(\mathtt{x} + \log T)^{3/4}}{T^{1/4}},
\end{aligned}$$

where $\xi^{\circ}_{\mathscr{I}} = \frac{1}{\sqrt{T}} \sum_{t \in \mathscr{I}} w_t Q^{-1} \mathbf{g}_t(\theta^*)$ and C does not depend on T and $\mathtt{x}$.

The papers Suvorikova and Spokoiny (2017) and Avanesov and Buzun (2016) apply the approach for change point detection. Following them, introduce the bootstrap test for change point s on the interval $\mathscr{I}$,

$$\begin{aligned}
T^{\circ}_{\mathscr{I},s} &= L^{\circ}_{A_s}(\widetilde{\theta}^{\circ}_{A_s}) + L^{\circ}_{B_s}(\widetilde{\theta}^{\circ}_{B_s}) - \sup\{L^{\circ}_{A_s}(\theta) + L^{\circ}_{B_s}(\theta + \widetilde{\theta}_{B_s} - \widetilde{\theta}_{A_s})\}, \\
T^{\circ}_{\mathscr{I}} &= \max_{s \in \mathscr{S}(\mathscr{I})} T_{\mathscr{I},s}.
\end{aligned}$$

Note, that here the shift $\widetilde{\theta}_{B_s} - \widetilde{\theta}_{A_s}$ is devoted to compensate the biases of the estimators $\widetilde{\theta}^{\circ}_{A_s}$ and $\widetilde{\theta}^{\circ}_{B_s}$ in the bootstrap world, which is not required in the original test. This test can further be used to simulate the critical values, since it's distribution conditioned on the data mimics the distribution of the original test $T_{\mathscr{I}}$ with high probability, as the following theorem states.

Theorem 2.1. *Suppose, that on an interval $\mathscr{I} \subset \{0, \dots, T\}$ the model satisfies 2.2-2.5 and 2.6. Suppose, that the set of break points satisfies for some $\alpha_0 > 0$*

$$\max_{s \in \mathscr{S}(\mathscr{I})} (|A_{\mathscr{I},s}|, |B_{\mathscr{I},s}|) \geq \alpha_0 |\mathscr{I}|. \tag{2.7}$$

Then, there are $C, c > 0$ that does not depend on $|\mathscr{I}|$, such that it holds with probability at least $1 - 1/|\mathscr{I}|$,

$$\sup_{z \in \mathbb{R}} |\mathsf{P}(T_{\mathscr{I}} > z) - \mathsf{P}^{\circ}(T^{\circ}_{\mathscr{I}} > z)| \lesssim C |\mathscr{I}|^{-c}.$$

The theorem justifies that the distribution of the bootstrap statistics $T^{\circ}_{\mathscr{I}}$ mimics the unknown distribution of the original statistics $T_{\mathscr{I}}$, so we can construct critical values for the change point test by simulating the bootstrap statistics:

$$z^{\circ}_{\mathscr{I}}(\alpha) = z^{\circ}_{\mathscr{I}}(\alpha; \mathbf{Y}) = \inf\{z \,:\, \mathsf{P}^{\circ}(T^{\circ}_{\mathscr{I}} > z) \leq \alpha\}, \tag{2.8}$$

is totally data-dependent and can be estimated via Monte-Carlo simulations with arbitrary precision (see Sections 5 for details). Given the theorem above, we can use these data-dependent critical values for the original test on the same data interval.

Corollary 2.1. *Under the assumptions of Theorem 2.1, we have*

$$|\mathsf{P}(T_{\mathscr{I}} > z^{\circ}_{\mathscr{I}}(\alpha)) - \alpha| \leq C |\mathscr{I}|^{-c},$$

where $C, c > 0$ do not depend on the interval length.

2.3 Localizing Multivariate CAViaR

Although time series should not be (globally) fitted by a parametric model with constant parameter, we assume that at each time point $t = 1, \dots, T$, there exists a historical interval $[t - m, t]$, over which the data process follows a parametric model, in our case equation (2.1). This local parametric assumption enables us to apply well-developed parametric estimation techniques in time series analysis. What is more, such an assumption includes the following scenarios as special cases: (i) the parameters are time-varying as the interval length changes over time and simultaneously (ii) our approach accounts for possible discontinuities and jumps in parameter coefficients as a function of time.

The essential idea of the proposed LMCR framework is to find the longest time series data interval over which the LMCR model can be "well" approximated by the parametric model. Therefore, the estimation procedure consists of two steps:

- for a time point of interest (usually latest available) select a historical interval that passes the homogeneity test described in the previous section;
- use the selected data interval for parameter estimation.

Interval Selection

The common way of selecting the homogeneous interval is as follows. To alleviate the computational burden, choose $(K+1)$ nested intervals of length $n_k = |\mathscr{I}_k|$, $k = 0, \ldots, K$, i.e., $\mathscr{I}_0 \subset \mathscr{I}_1 \subset \cdots \subset \mathscr{I}_K$. Interval lengths are usually taken to be geometrically increasing with $n_k = \lceil n_0 c^k \rceil$, where $c > 1$ is slightly greater than one, so that in the worst case one only neglects a small proportion of unknown homogeneous interval. We assume that the initial interval $\mathscr{I}_0$ is small enough, so that the model parameters are constant within this interval.

Further, we conduct a sequential testing procedure. For each $k = 1, \ldots, K$ we want to test the homogeneity of the parameter over interval $\mathscr{I}_k$ against the alternative of homogeneity over interval $\mathscr{I}_{k-1}$. By our assumption $\mathscr{I}_0$ is homogeneous. The resulting interval of homogeneity would then be the last before the first one rejected. Therefore, for each such $k = 1, \ldots, K$ we choose a set of breaking points $\mathscr{S}_k = \mathscr{I}_k \setminus \mathscr{I}_{k-1}$ outside of the interval that we already tested. Using the testing procedure from Section 2.2 we reject the kth interval, if

$$\max_{s \in \mathscr{S}_k} T_s > \mathfrak{z}^{\circ}_{\mathscr{I}_k}(\alpha),$$

where $\mathfrak{z}^{\circ}_{\mathscr{I}_k}(\alpha)$ is generated through multiplier bootstrap (2.8). Observe that if the model is homogeneous on a historical interval $[t-n^*, t]$, then due to Corollary 2.1 we will accept homogeneity of each interval $\mathscr{I}_k = [t-n_k, t]$ with $n_k \le n^*$ with high probability. If an interval $\mathscr{I}_k$ remains homogeneous, the estimator $\widetilde{\theta}_{\mathscr{I}_k}$ has small bias, while the variance decreases with growing number of observations, according to Theorem 2.2. The least variance, therefore, corresponds to the largest found interval of homogeneity, and the final estimator reads as

$$\widehat{\theta} = \widetilde{\theta}_{\mathscr{I}_\kappa}, \qquad \kappa = \max\{k \,:\, \mathscr{I}_k \text{ is not rejected against } \mathscr{I}_{k-1}\}.$$

This finishes the second step of our LMCR estimator. In the next two sections we analyse the proposed procedure numerically.

2.4 Simulation

In this section we study the effectiveness of our adaptive approach in detecting the structure breaks in numerical analysis. Following the setup of WKM and the simulation study in Gerlach et al. (2011) and Hong et al. (2009), we generate the data time series using a two-variate GARCH process:

$$\begin{aligned} \sigma_{1t} &= \widetilde{\beta}_{11}\sigma_{1t-1} + \widetilde{\beta}_{12}\sigma_{2t-1} + \widetilde{\gamma}_{11}|y_{1t-1}| + \widetilde{\gamma}_{12}|y_{2t-1}| + \widetilde{c}_1 \\ \sigma_{2t} &= \widetilde{\beta}_{21}\sigma_{1t-1} + \widetilde{\beta}_{22}\sigma_{2t-1} + \widetilde{\gamma}_{21}|y_{1t-1}| + \widetilde{\gamma}_{22}|y_{2t-1}| + \widetilde{c}_2 \\ Y_{it} &= \sigma_{it}\varepsilon_{it}, \qquad \varepsilon_{it} \sim \mathrm{N}(0,1) \ \text{ i.i.d.} \qquad i = 1,2 \end{aligned} \tag{2.9}$$

Denote the parameter set $\widetilde{\theta} = (\widetilde{\beta}_{ij}, \widetilde{\gamma}_{ij}, \widetilde{c}_i)$ where $i, j = 1,2$.

Note that at a given quantile level τ, the quantile process $q_{it}(\tau) = \mathrm{Quant}_\tau(Y_{it}| \mathscr{F}_{t-1})$ satisfies $q_{it}(\tau) = \Phi^{-1}(\tau)\sigma_{it}$, where $\Phi^{-1}(\tau)$ is the quantile function of the standard normal distribution. Therefore, the following recurrent equation takes place

$$\begin{aligned} q_{1t}(\tau) &= \beta_{11}q_{1t-1}(\tau) + \beta_{12}q_{2t-1}(\tau) + \gamma_{11}|y_{1t-1}| + \gamma_{12}|y_{2t-1}| + c_1 \\ q_{2t}(\tau) &= \beta_{21}q_{1t-1}(\tau) + \beta_{22}q_{2t-1}(\tau) + \gamma_{21}|y_{1t-1}| + \gamma_{22}|y_{2t-1}| + c_2, \end{aligned} \tag{2.10}$$

where the parameter $\theta_\tau = (\beta_{ij}, \gamma_{ij}, c_i)_{i,j=1,2}$ consists of ten coefficients $\beta_{ij} = \widetilde{\beta}_{ij}$ and $\gamma_{ij} = \Phi^{-1}(\tau)\widetilde{\gamma}_{ij}$, $c_i = \Phi^{-1}(\tau)\widetilde{c}_i$ for $i, j = 1,2$.

For simulations we consider a time series $(Y_{it})_{t=1}^{500}$ with the initial variances $\sigma_{i1} = 1$ and parameters

$$\begin{aligned} \theta_{left} =& (0.5,0,0,0.5,0,0.2,0.2,0,0.5,0.5), \\ \theta_{right} =& (-0.5,0,0,0.5,0,0.2,0.2,0,0.5,0.5), \end{aligned}$$

so that before the break $t \leq s = 250$ the time series satisfies (2.9) with the parameter θ_{left} and after the break with θ_{right}. For each time point with step 20 (i.e. 500, 480, 460, and so on) we test a nested sequence of intervals $I_0 \subset I_1 \subset \cdots \subset I_K$ with lengths $n_k = \lceil c^k |I_0| \rceil$, which we take with $K = 9$, $|I_0| = 60$ and $c = 1.25$. The considered lengths of intervals are therefore,

$$\{60, 72, 87, 104, 125, 150, 180, 215, 258\}.$$

The results for choosing the interval length are presented on the Figure 2.1. On Figures 2.2, 2.3 we show estimated conditional quantiles $\widehat{q}_{it}$ based on the observations available at a point $t-1$, using the corresponding selected homogeneity intervals.

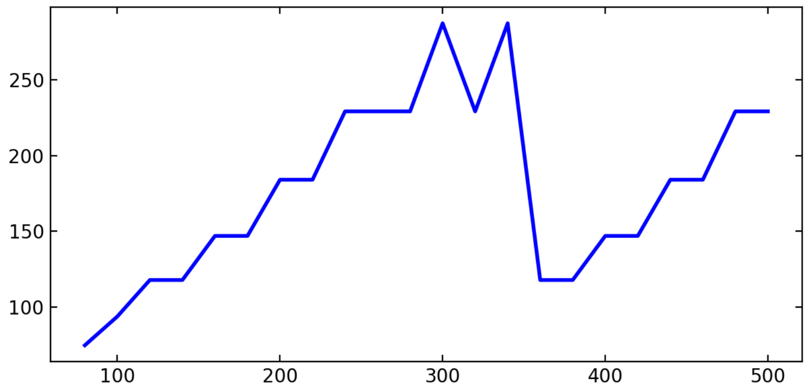


Figure 2.1 Selected length of homogeneous intervals for timepoints 80 to 500 with step 20.

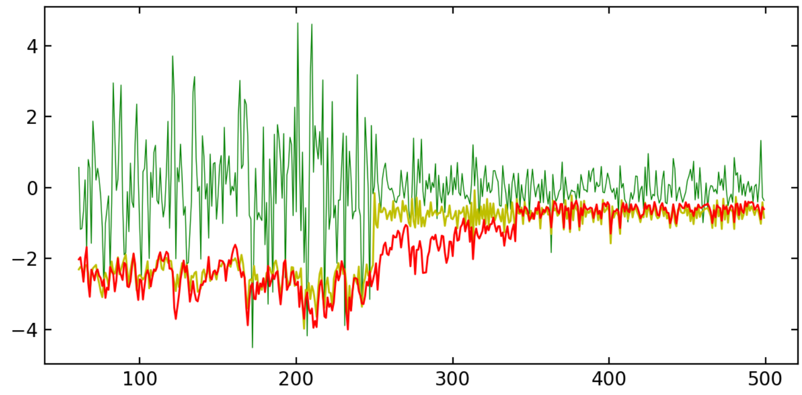


Figure 2.2 LMCR's predicted quantile one step ahead (red), actual quantile (yellow) and the original simulated time series (green) for $i = 1$ in (2.10).

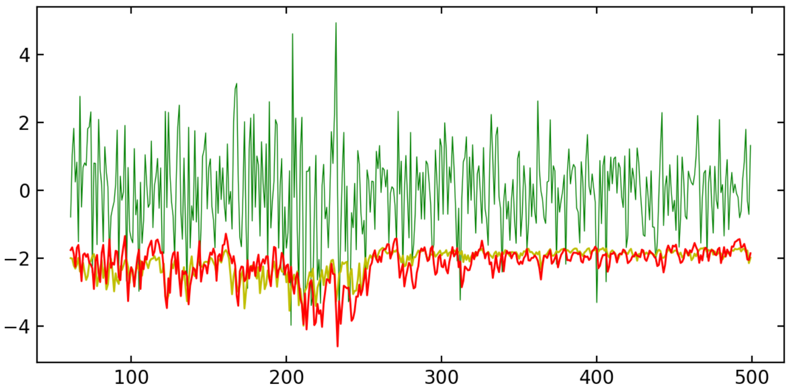


Figure 2.3 LMCR's predicted quantile one step ahead (red), actual quantile (yellow) and the original simulated time series (green) for $i = 2$ in (2.10).

Localizing_Multivariate_CAViaR

Numerical implementation

The optimization problem (2.3) is computationally involved. We deal with a highly non-concave target function, that may even have various local maxima. Indeed, the quantile functions (2.2) are polynomials of a multivariate parameter, with the total degree growing up to the number of observations. Notice also that the equation (2.1) is a simple Recurrent Neural Network with a linear activation function and one can use software developed specifically for fitting neural networks. We choose to use python's Keras package with TensorFlow backend. The package exploits gradient descent, and the procedure is well optimized. These simulation codes are available at github.com/QuantLet/mvcaviar. In addition, the following application results and the corresponding MATLAB programming codes can be found in the folder github.com/QuantLet/LMVCAViaR. All these are available at quantlet.de.

2.5 Application

2.5.1 Data and Parameter Dynamics

We consider two stock markets, namely, the S&P 500 and DAX series. Daily index returns are obtained from Datastream and our data cover the period from 3 January 2005 to 29 December 2017, in total 3390 trading days. The daily returns evolve similarly across the selected markets and all present relatively large variations during the financial crisis period from 2008–2010, see Figure 2.4. Although the return time series exhibit nearly zero-mean with slightly pronounced skewness values, all present comparatively high kurtosis, see Table 2.1 that collects the summary statistics.

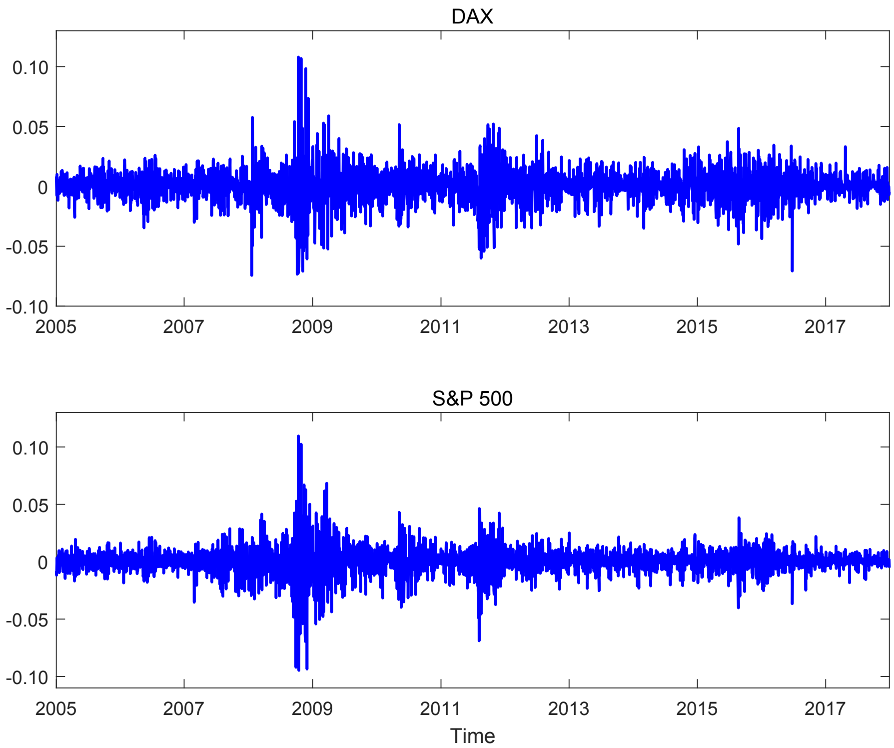


Figure 2.4 Selected index return time series from 3 January 2005 to 29 December 2017 (3390 trading days).

LMVCAViaR_return_plot

We utilize model (2.10) in the study of the selected (daily) stock market indices. We firstly consider different interval lengths (e.g., 60 and 500 observations) and analyze the corresponding estimates. One may observe a relatively large variability of the estimated parameters while fitting the model over short data intervals and vice versa. The time-variation of the parameter are presented here via two quantile levels, namely $\tau = 0.01$ and $\tau = 0.05$.

Parameter estimates are indeed more volatile when fitting the MV-CAViaR over shorter intervals (60 days), see e.g. Figures 2.5 and 2.6. More precisely, we display the estimated MV-CAViaR parameters $\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\beta}_{22}$ in model (2.10) in rolling window exercises from 1 January 2007 to 29 December 2017. The upper (lower) panel at each figure shows the estimated parameter values if 60 (500) observations are included in the respective window.

Index	Mean	Median	Min	Max	Std	Skew.	Kurt.
S&P 500	0.0002	0.0003	-0.0947	0.1096	0.0121	-0.3403	14.6949
DAX	0.0003	0.0007	-0.0743	0.1080	0.0137	-0.0406	9.2297

Table 2.1 Descriptive statistics for the selected index return time series from 3 January 2005 to 29 December 2017 (3390 trading days): mean, median, minimum (Min), maximum (Max), standard deviation (Std), skewness (Skew.) and kurtosis (Kurt.).

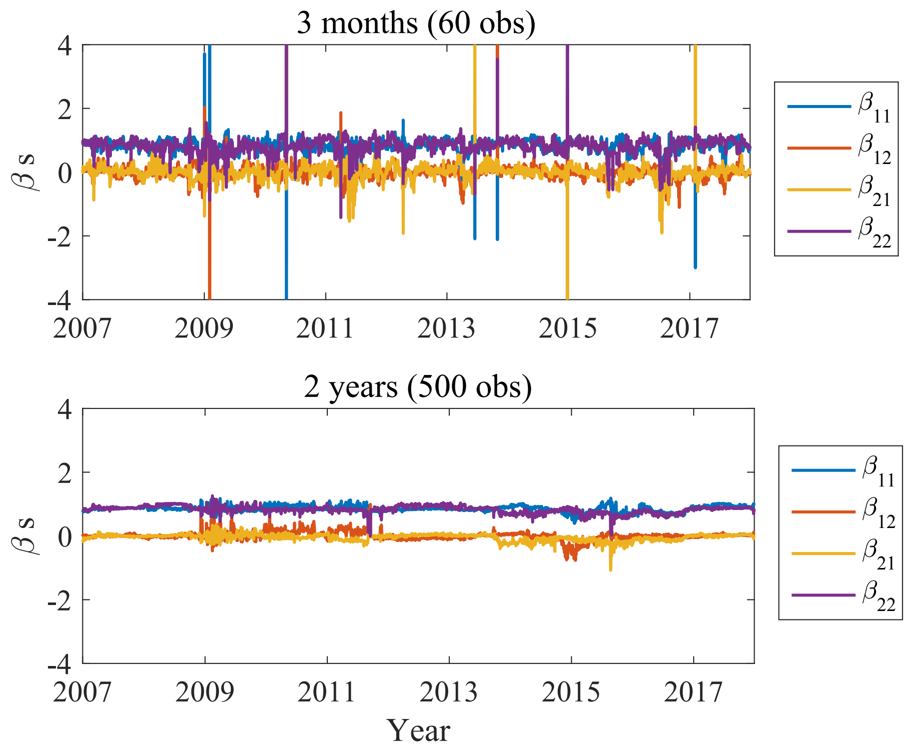


Figure 2.5 Estimated parameters $\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\beta}_{22}$ at quantile level $\tau = 0.05$ for the selected two stock markets from 1 January 2007 to 29 December 2017, with 60 (upper panel) and 500 (lower panel) observations used in the rolling window exercises.

LMVCAViaR_estimate_rolling

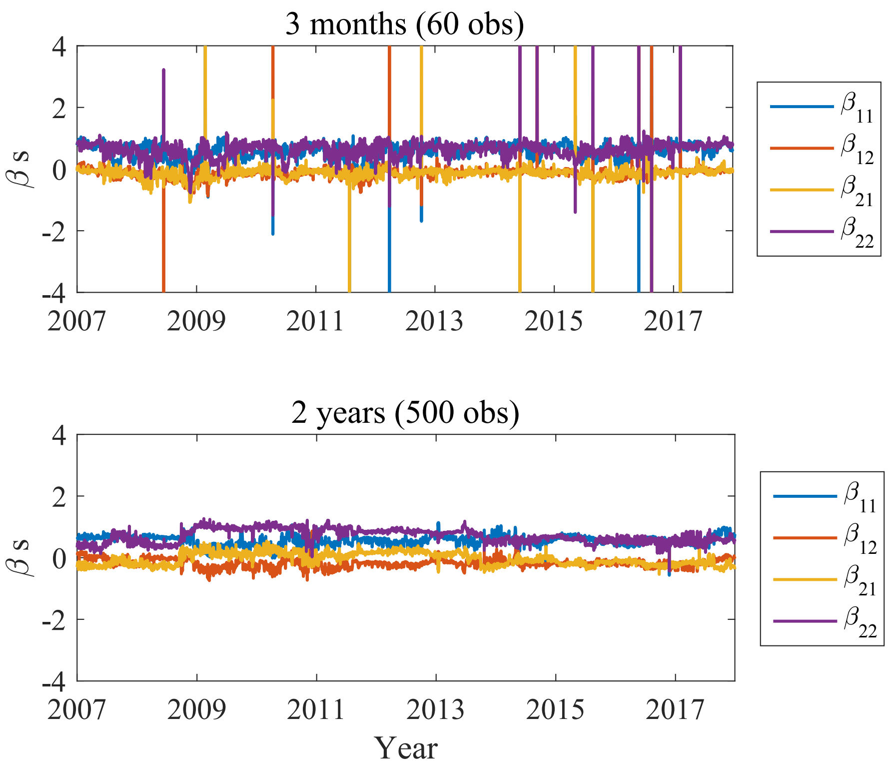


Figure 2.6 Estimated parameters $\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\beta}_{22}$ at quantile level $\tau = 0.01$ for the selected two stock markets from 1 January 2007 to 29 December 2017, with 60 (upper panel) and 500 (lower panel) observations used in the rolling window exercises.

LMVCAViaR_estimate_rolling

Key empirical results from the presented fixed rolling window exercise can be summarized as follows: (a) there exists a trade-off between the modeling bias and parameter variability across different estimation setups, (b) the characteristics of the time series of estimated parameter values as well as the estimation quality results demand the application of an adaptive method that successfully accommodates time-varying parameters, (c) data intervals covering 60 to 500 observations may provide a good balance between the bias and variability. Motivated by these findings, we now turn to LMCR.

We exactly follow the steps as described in Section 2.2 to implement LMCR in the application. In line with the aforementioned empirical results, we select $(K+1) = 13$ intervals, starting with 60 observations (three months) and ending with 500 observations (two trading years), i.e., we consider the set

$$\{60, 75, 94, 118, 148, 185, 231, 289, 361, 451, 500\}$$

with the coefficient $c = 1.25$ in accordance with the literature. In addition, we assume the model parameters are constant within the initial interval $I_0 = 60$.

Meanwhile, we use the initial two-year time series, i.e. from 3 January 2005 to 30 December 2006, as the training sample to simulate the critical values. We exactly follow the procedure described in Section 2.2.1 to operate the simulation. We set two cases of the tuning parameter: the conservative case $\alpha = 0.8$ and the modest case $\alpha = 0.9$ to choose the critical values. We present the empirical results in the next section.

2.5.2 Results

LMCR accommodates and reacts to structural changes. From the fixed rolling window exercise in subsection 2.5.1 one observes time-varying parameter characteristics while facing the trade-off between parameter variability and the modelling bias. How to account for the effects of potential market changes on the tail risk based on the intervals of homogeneity? In the application, we employ LMCR to estimate the tail risk exposure as well as to analyze the cross-sectional spillover effects between the two selected stock markets. Using the time series of the adaptively selected interval length, one can trace out the dynamic tail risk spillovers and identify the distinct roles in risk transmissions.

A. Homogeneous Intervals

The interval of homogeneity in tail quantile dynamics is obtained here by the LMCR framework for the time series of DAX and S&P 500 returns. Using the sequential local change point detection test, the optimal interval length is considered at two quantile levels, namely, $\tau = 0.01$ and $\tau = 0.05$, see Figure 2.8 and 2.7. All figures present the estimated lengths of the interval of homogeneity in trading days using the selected stock market indices from 1 January 2007 to 29 December 2017. The upper panel depicts the conservative risk case $\alpha = 0.8$, whereas the lower panel denotes the modest risk case $\alpha = 0.9$.

In a similar way, the intervals of homogeneity are slightly shorter in the conservative risk case $\alpha = 0.8$, as compared to the modest risk case $\alpha = 0.9$. The average daily selected optimal interval length supports this, see, e.g., Table 2.2. The results are presented for the selected quantile levels at the conservative and modest risk cases, $\alpha = 0.8$ and $\alpha = 0.9$, respectively. In general the average lengths of selected intervals range between 7-10 months of daily observations across different markets. At quantile levels $\tau = 0.05$, the intervals of homogeneity are slightly larger than the intervals at $\tau = 0.01$.

	$\alpha = 0.8$	$\alpha = 0.9$
$\tau = 0.05$	159	231
$\tau = 0.01$	143	171

Table 2.2 Mean value of the adaptively selected intervals. Note: the average number of trading days of the adaptive interval length is provided for the DAX and S&P 500 market indices at quantile levels, $\tau = 0.05$ and $\tau = 0.01$, and the conservative ($\alpha = 0.80$) and the modest ($\alpha = 0.90$) risk case.

LMVCAViaR_adaptive_estimation_length

B. One-Step-Ahead Forecasts of Tail Risk Exposure

Based on LMCR, one may directly estimate dynamic tail risk exposure. The tail risk at smaller quantile level is relatively lower than risk at higher levels, see, e.g., Figure 2.9. Here the estimated quantile risk exposure for the two stock market indices from 1 January 2007 to 29 December 2017 is displayed for two quantile levels, $\tau = 0.01$ and $\tau = 0.05$. The left panel represents the conservative risk case $\alpha = 0.8$ results, whereas the right panel considers the modest risk case $\alpha = 0.9$. The latter leads on average to slightly lower variability, as

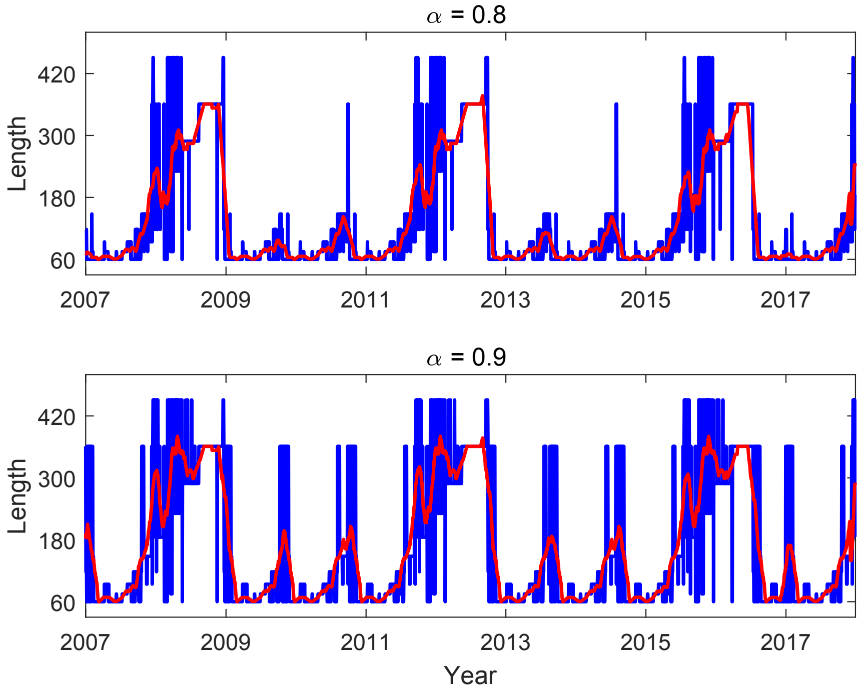


Figure 2.7 Estimated length of the interval of homogeneity in trading days for the selected stock markets from 1 January 2007 to 29 December 2017 for the conservative (upper panel, $\alpha = 0.8$) and the modest (lower panel, $\alpha = 0.9$) risk cases. The quantile level equals $\tau = 0.01$. The red line denotes one-month smoothed values.

LMVCAViaR_adaptive_estimation_length LMVCAViaR_adaptive_estimation_001

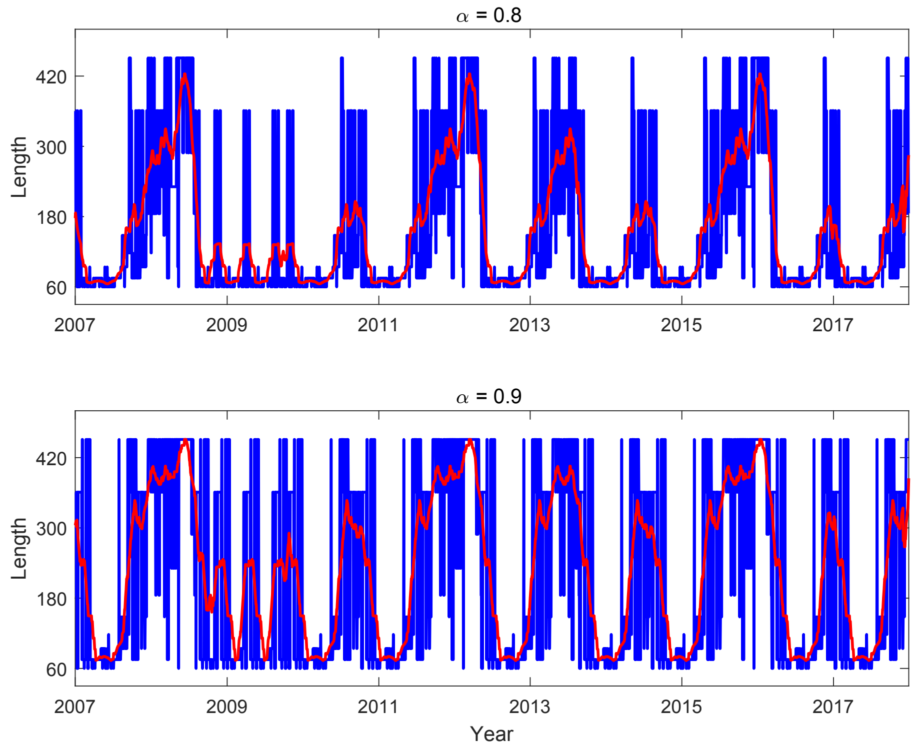


Figure 2.8 Estimated length of the interval of homogeneity in trading days for the selected stock markets from 1 January 2007 to 29 December 2017 for the conservative (upper panel, $\alpha = 0.8$) and the modest (lower panel, $\alpha = 0.9$) risk cases. The quantile level equals $\tau = 0.05$. The red line denotes one-month smoothed values.

LMVCAViaR_adaptive_estimation_length LMVCAViaR_adaptive_estimation_005 compared to the conservative risk case which results in marginally shorter homogeneity intervals.

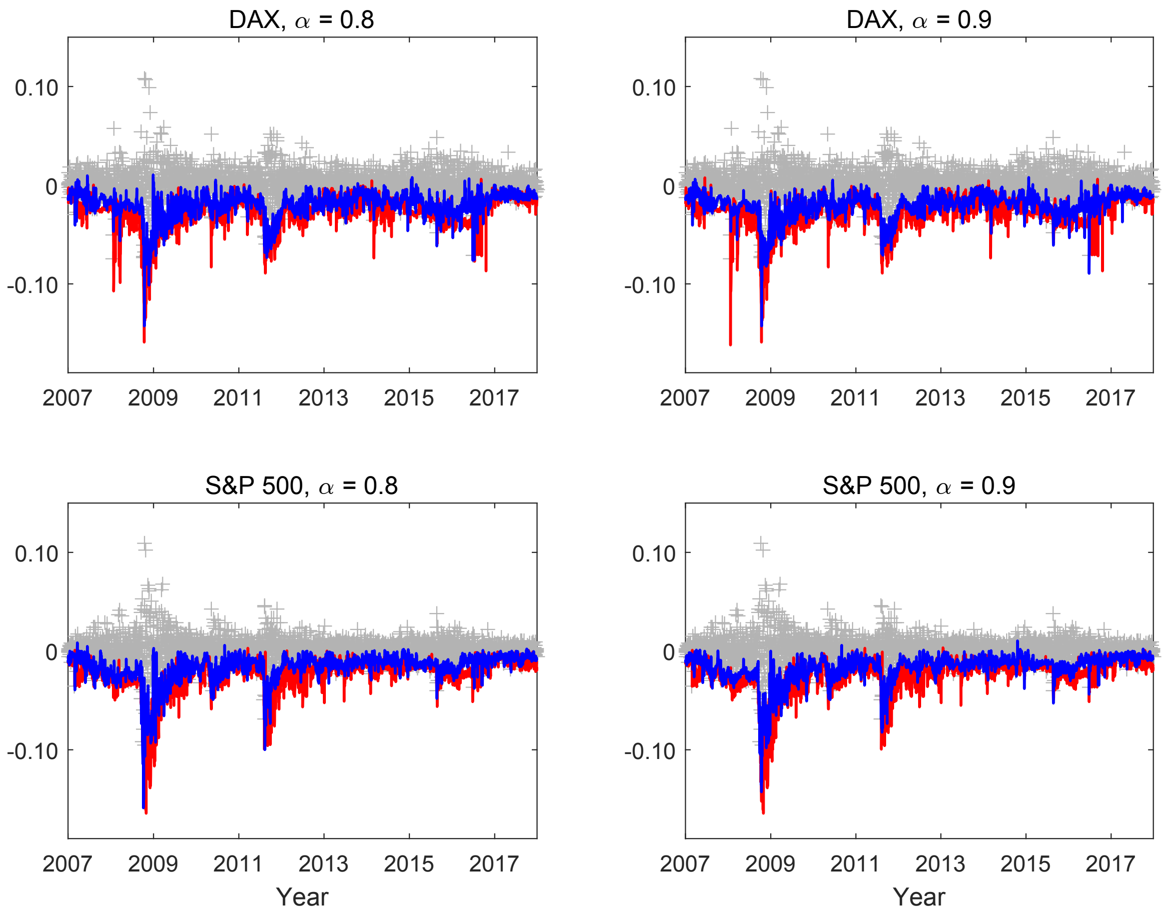


Figure 2.9 One-step ahead forecasts of quantile risk exposure at level $\tau = 0.05$ (blue) and $\tau = 0.01$ (red) for return time series of DAX and S&P 500 indices (grey points) from 1 January 2007 to 29 December 2017. The left panel shows results of the conservative risk case $\alpha = 0.8$ and the right panel depicts results of the modest risk case $\alpha = 0.9$.

LMVCAViaR_adaptive_quantile

C. Time-Varying Coefficient Estimates

The transitions among the financial markets are directly revealed by the cross-sectional coefficients, see Adams et al. (2014). Here we take the dynamics of the two coefficients, β_{12} and β_{21}, as representations of spillover effects between S&P 500 and DAX. Figure 2.10 and 2.11 plot the dynamics of spillover effects from S&P 500 to DAX, β_{12} and the ones from

DAX to S&P 500, β_{21}. The upper (lower) panel represent the case of quantile level $\tau = 0.01$ ($\tau = 0.05$). The blue lines show results of the conservative risk case $\alpha = 0.8$ and the red lines depict results of the modest risk case $\alpha = 0.9$.

Moreover, it shows that the cross-sectional coefficient β_{12} presents larger and more volatile dynamics compared with the coefficient β_{21} for both quantile levels $\tau = 0.01$ and $\tau = 0.05$. The shifting of the risk spillovers from US market to German market tend to be more intensive, especially during the unstable market period, e.g. the 2008 financial crisis period and the 2012 European sovereign debt crisis. Hence, compared with the spillovers from DAX to S&P 500, the US market appears to play dominate role in risk transmissions of shocks to DAX indice, especially in volatile time.

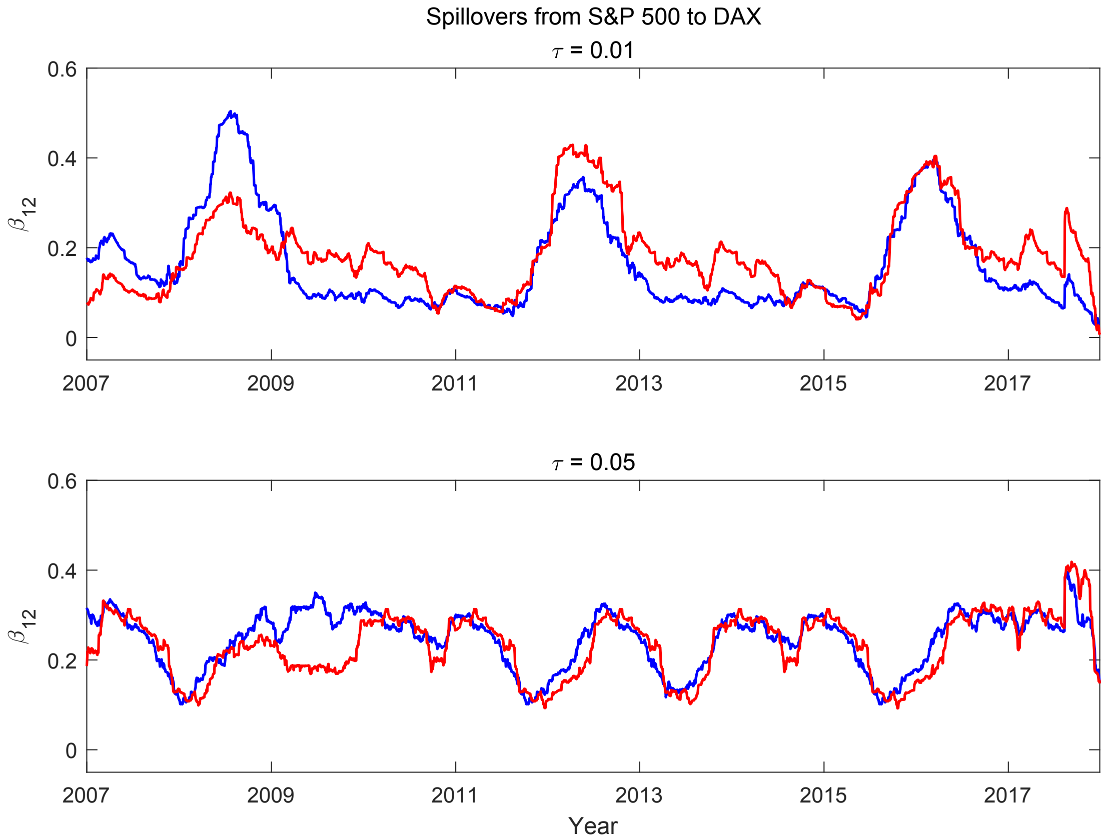


Figure 2.10 Time-varying coefficients β_{12} at quantile level $\tau = 0.01$ (upper panel) and $\tau = 0.05$ (lower panel) for return time series of DAX and S&P 500 indices from 1 January 2007 to 29 December 2017. The blue lines show results of the conservative risk case $\alpha = 0.8$ and the red lines depict results of the modest risk case $\alpha = 0.9$.

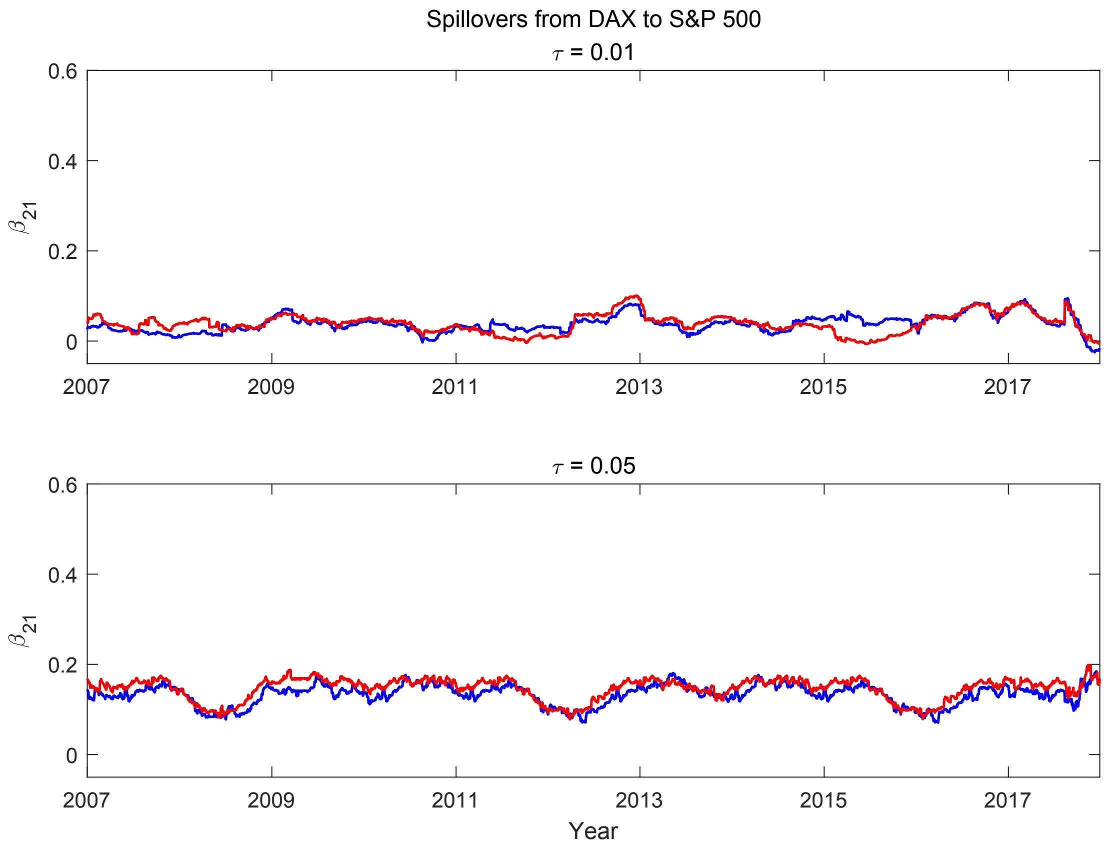


Figure 2.11 Time-varying coefficients β_{21} at quantile level $\tau = 0.01$ (upper panel) and $\tau = 0.05$ (lower panel) for return time series of DAX and S&P 500 indices from 1 January 2007 to 29 December 2017. The blue lines show results of the conservative risk case $\alpha = 0.8$ and the red lines depict results of the modest risk case $\alpha = 0.9$.

2.6 Conclusion

The cross-sectional tail risk dependence among financial markets are time-varying and LMCR is constructed to cope with this challenge in evaluating the risk contagion. A local adaptive approach assumes that at any given point of time there is a historical interval of observations over which the time series follows a parametric model. By utilizing a local change point detection procedure, one can sequentially determine the interval of homogeneity over which the time series behavior can be approximated described by a fixed parameter. LMCR adaptively estimates the tail risk transmission by relying on the longest detected interval of homogeneity. The corresponding statistical properties of this method are successfully derived.

A comprehensive simulation study supports the effectiveness of our approach in detecting structural changes in multivariate tail risk estimation. When setting the quantile levels at $\tau = 0.05$ and $\tau = 0.01$ in a application of stock market indices DAX and S&P 500, the dynamic tail risk measures are successfully obtained. In addition, the developed approach permits a delineation of the shifting tail risk spillover effects. We find that the US market tends to play prominent role in risk transmissions of shocks to German market, especially in volatile times.

2.7 Proofs

Without loss of generality in Sections 2.7.1–2.7.4 we assume, that the interval of interest is the whole observed data set, i.e. $\mathscr{I} = \{0, \dots, T\}$. For this reason we neglect the index "$\mathscr{I}$" where applies, for instance, $L(\widetilde{\theta})$ instead of $L_{\mathscr{I}}(\widetilde{\theta}_{\mathscr{I}})$.

2.7.1 Proof of Lemma 2.1

Denote,

$$\tilde{\mathbf{g}}_t(\theta) = \mathbf{g}_t(\theta) - \sum_i \nabla q_{it}(\theta^*)\, \mathbf{I}^c[Y_{it} \leq q_{it}(\theta)],$$

where for $\mathscr{F}_{t-1}$–measurable Z we set $\mathbf{I}^c[Y_{it} \leq Z] = \mathbf{I}[Y_{it} \leq Z] - \mathsf{P}(Y_{it} \leq Z | \mathscr{F}_{t-1})$. Since $q_{it}(\theta)$ are $\mathscr{F}_{t-1}$–measurable, we obviously have $\mathsf{E}\tilde{\mathbf{g}}_t(\theta) = \lambda_t(\theta)$. For any two $\theta, \theta' \in \Theta$ consider the decomposition,

$$\begin{aligned}\mathbf{g}_t(\theta) - \mathbf{g}_t(\theta') =& \sum_i \{\nabla q_{it}(\theta) - \nabla q_{it}(\theta')\}\psi_{\tau_i}(Y_{it} - q_{it}(\theta)) \\ &+ \sum_i \nabla q_{it}(\theta^*)\left\{\mathsf{P}[Y_{it} \le q_{it}(\theta)|\,\mathscr{F}_{it}] - \mathsf{P}[Y_{it} \le q_{it}(\theta')|\,\mathscr{F}_{it}]\right\} \\ &+ \sum_i \nabla q_{it}(\theta^*)\left\{\mathbf{I}^c[Y_{it} \le q_{it}(\theta)] - \mathbf{I}^c[Y_{it} \le q_{it}(\theta')]\right\},\end{aligned}$$

and, similarly, the difference $\tilde{\mathbf{g}}_t(\theta) - \tilde{\mathbf{g}}_t(\theta^*)$ has only two first terms in this decomposition. In the proof of Theorem 2 of White et al. (2015) it is shown, that with Assumption 2.3

$$\|\tilde{\mathbf{g}}_t(\theta) - \tilde{\mathbf{g}}_t(\theta')\| \le D_2(np + f_0 D_1)\|\theta - \theta'\|.$$

Let us fix some unit $\gamma \in \mathbb{R}^p$ and apply Theorem 1 of Merlevède et al. (2009) to the sum $\sum_t \gamma^\top\{\tilde{\mathbf{g}}_t(\theta) - \tilde{\mathbf{g}}_t(\theta')\}$. Since by Assumption 2.4 it holds $\alpha(k) \le \exp(-ck)$, we have a Hoeffding-type inequality for each $\mathrm{x} \ge 0$,

$$\gamma^\top\left\{\sum_t \tilde{\mathbf{g}}_t(\theta) - \lambda_t(\theta) - \tilde{\mathbf{g}}_t(\theta') + \lambda_t(\theta')\right\} > C_1\|\theta - \theta'\|(\sqrt{\mathrm{x}T} + \mathrm{x}\log^2 T) \tag{2.11}$$

with probability $\ge 1 - C_2 e^{-\mathrm{x}}$, where C_1 and C_2 only depend on γ. Further we apply Theorem 2.2.27 of Talagrand (2014a) to get for any $\mathrm{x} \ge 0$

$$\mathsf{P}\left(\sup_{\theta\in\Theta\,:\,\|\theta-\theta^*\|\le\mathtt{r}}\left\|\sum_t \tilde{\mathbf{g}}_t(\theta) - \lambda_t(\theta) - \tilde{\mathbf{g}}_t(\theta') + \lambda_t(\theta')\right\| > LA(\mathtt{r},\mathrm{x})\right) \le LC_2 e^{-\mathrm{x}},$$

where $A(\mathtt{r},\mathrm{x}) = \sqrt{T}\gamma_2(\mathtt{r}B_1, \|\cdot\|)\sqrt{\mathrm{x}} + (\log^2 T)\gamma_1(\mathtt{r}B_1,\|\cdot\|)\mathrm{x}$, with L being a generic constant, B_1 is a unit ball in $\mathbb{R}^p$, and $\gamma_{1,2}(T,\|\cdot\|)$ are Talagrand gamma-functionals, precisely, see Definition 2.2.18 in Talagrand (2014a). In the case of finite dimensional space, we have $\gamma_{1,2}(\mathtt{r}B_1(0),\|\cdot\|) \le \mathtt{r}C$, where $C = C(p)$ only depends on the dimension. We therefore can rewrite the above inequality,

$$\mathsf{P}\left(\sup_{\theta\in\Theta\,:\,\|\theta-\theta^*\|\le\mathtt{r}}\left\|\sum_t \tilde{\mathbf{g}}_t(\theta) - \lambda_t(\theta) - \tilde{\mathbf{g}}_t(\theta') + \lambda_t(\theta')\right\| > C\mathtt{r}(\sqrt{\mathrm{x}T} + \mathrm{x}\log^2 T)\right) \le e^{-\mathrm{x}},$$

where C only depends on n and γ, and $\mathrm{x} \ge 1$.

Consider a δ-net $\{\theta_1,\ldots,\theta_N\}$ of the set $\Theta_0(\mathtt{r})$, so that for each $\theta \in \Theta_0(\mathtt{r})$ there is $j = 1..N$ with $\|\theta - \theta_j\| \le \delta$. It is known, that there is such a set with $\log N \le Cp\log\frac{\mathtt{r}}{\delta}$

elements. By Bernstein-type inequality, Theorem 2 in Merlevède et al. (2009), it holds

$$\left\| \sum_t \sum_i \nabla q_{it}(\theta^*)(\mathbf{I}^c[Y_{it} \le q_{it}(\theta_k)] - \mathbf{I}^c[Y_{it} \le q_{it}(\theta^*)]) \right\| \le C\{\sqrt{\mathtt{r}T}\sqrt{\mathtt{x}+\log N} + (\log T)^2(\mathtt{x}+\log N)\},$$

uniformly for all $k = 1,\dots,N$ with probability at least $1-e^{-\mathtt{x}}$, and the constant only depend on n,γ. Here we use the fact that the terms $\mathbf{I}^c[Y_{it} \le q_{it}(\theta)]$ are centred conditioned on $\mathscr{F}_{t-1}$, while $\nabla q_{it}(\theta)$ are $\mathscr{F}_t$ measurable.

Furthermore, taking into account part (iii) of Assumption 2.4 we can use Theorem 5.2 from Boucheron et al. (2005a) to get that for any $i = 1,\dots,n$

$$|\{t : \varepsilon_{it} \in [a,b]\}| \le T f_0(b-a) + C\sqrt{T f_0(b-a)\mathtt{x}} + C\mathtt{x}$$

with probability at least $1-4e^{-\mathtt{x}}$ uniformly over all intervals, with some universal constant C. By definition, for any $\theta \in \Theta_0(\mathtt{r})$ there is some k such that $|g_{it}(\theta) - g_{it}(\theta_k)| \le D_1\delta$ for each i,t. Therefore, the amount of indices i,t, for which the values of $\mathbf{I}[Y_{it} - q_{it}(\theta)]$ and $\mathbf{I}[Y_{it} - q_{it}(\theta_k)]$ differ is bounded by $C(T\delta + \sqrt{T\delta\mathtt{x}} + \mathtt{x})$, constant C does not depend on $T,\mathtt{x},\mathtt{r}$ and δ. We come to the conclusion, that choosing $\delta = \mathtt{r}T^{-1/2}$, on the intersection of the events listed above it holds,

$$\left\| \sum_t \sum_i \nabla q_{it}(\theta^*)\{\mathbf{I}[Y_{it} \le q_{it}(\theta)] - \mathbf{I}[Y_{it} \le q_{it}(\theta_k)]\} \right\| \lesssim T^{1/2}\mathtt{r} + \sqrt{T^{1/2}\mathtt{r}\mathtt{x}} + \mathtt{x}.$$

Putting the inequalities together we get the result.

2.7.2 Proof of Proposition 2.1

The claim follows directly from a slightly flexible version, that we are using for the consistency of bootstrap estimator as well.

Lemma 2.2. *Let assumptions 2.1–2.5 hold on the interval $\mathscr{I}$. Then there are $T_0, a_0 > 0$ such that whenever $|\mathscr{I}| \ge T_0$, $a \le a_0$ and $\mathtt{x} \le |\mathscr{I}|$ the following implication takes place with probability $\ge 1-6e^{-\mathtt{x}}$. Each $\theta \in \Theta$ that satisfies,*

$$L_{\mathscr{I}}(\theta) - L_{\mathscr{I}}(\theta^*) \ge -|\mathscr{I}|a$$

satisfies as well

$$\|\theta - \theta^*\| \le \sqrt{a/b} + C_0 \sqrt{\frac{\mathtt{x} + \log|\mathscr{I}|}{|\mathscr{I}|}},$$

where b, C_0 do not depend on $|\mathscr{I}|$ and $\mathtt{x}$.

First, we present a uniform bound for the score. Similar to (2.11) it holds $\|\nabla\zeta(\theta^*)\| \le C(\sqrt{\mathtt{x}T} + \mathtt{x}\log^2 T)$ with probability $\ge 1 - e^{-\mathtt{x}}$, while by Lemma 2.1 we have with probability $\ge 1 - e^{-\mathtt{x}}$, that

$$\sup_{\theta \in \Theta_0} \|\nabla\zeta(\theta) - \nabla\zeta(\theta^*)\| \le C(\sqrt{T}\sqrt{\mathtt{x} + \log T} + \mathtt{x}\log^2 T),$$

using the fact that the set Θ_0 is bounded. Using a simple triangle inequality we have,

$$\|\nabla\zeta_{\mathscr{I}}(\theta)\| \le C(\sqrt{T}\sqrt{\mathtt{x} + \log T} + \mathtt{x}\log^2 T) \tag{2.12}$$

with probability $\ge 1 - 2e^{-\mathtt{x}}$ uniformly for each $\theta \in \Theta_0$, with C not depending on $T, \mathtt{x}$.

Next we present a technical lemma, that shows quadratic deviation of the expectation of log-likelihood in the neighbourhood of true parameter. The resulting inequality is akin to condition $(\mathscr{L}_r)$ of Spokoiny (2017).

Lemma 2.3. *Suppose, 2.1–2.3 and 2.5 hold. Then, there are $\mathtt{r}_0, b > 0$ that do not depend on $|\mathscr{I}|$, such that for each $\theta \in \Theta$ satisfying $\|\theta - \theta^*\| \ge \mathtt{r}$ it holds $\mathsf{E}L_{\mathscr{I}}(\theta) - \mathsf{E}L_{\mathscr{I}}(\theta^*) \le -b|\mathscr{I}|(\mathtt{r}^2 \wedge \mathtt{r}_0^2)$.*

The proof of this lemma is postponed to Section 2.7.6.

Proof of Lemma 2.2. By (2.12) we have for $\mathtt{x} \le |\mathscr{I}|$,

$$\begin{aligned}
\frac{1}{|\mathscr{I}|}\mathsf{E}L_{\mathscr{I}}(\theta) - \frac{1}{|\mathscr{I}|}\mathsf{E}L_{\mathscr{I}}(\theta^*) \ge& L_{\mathscr{I}}(\theta) - L_{\mathscr{I}}(\theta^*) - \|\theta - \theta^*\| \sup_{\theta \in \Theta} \|\nabla\zeta_{\mathscr{I}}(\theta)\| \\
\ge& -a - C_2\|\theta - \theta^*\||\mathscr{I}|^{-1/2}\sqrt{\mathtt{x} + \log|\mathscr{I}|} \\
\ge& -a_0 - C_2 R|\mathscr{I}|^{-1/2}\sqrt{\mathtt{x} + \log|\mathscr{I}|}
\end{aligned}$$

with probability at least $1 - 2e^{-\mathtt{x}}$. By Lemma 2.3 this implies,

$$b\|\theta - \theta^*\|^2 \le a + C_2\|\theta - \theta^*\||\mathscr{I}|^{-1/2}\sqrt{\mathtt{x} + \log|\mathscr{I}|},$$

and it is left to notice that $x^2 \le \alpha + \beta x$ implies $x \le \sqrt{\alpha} + \beta$. Additionally, $L(\widetilde{\theta}) \ge L(\theta^*)$ pointwise, thus the deviation bound for the estimator takes place. $\square$

2.7.3 Proof of Proposition 2.2

First of all, by Proposition 2.1 it holds with probability $\ge 1 - 7e^{-\mathtt{x}}$, that $\|\widetilde{\theta} - \theta^*\| \le \mathtt{r}_0 = C_0\sqrt{T^{-1}(\mathtt{x} + \log T)}$. Applying Lemma 2.1 with this radius, we get that with probability $\ge 1 - 13e^{-\mathtt{x}}$ additionally this holds for each $\theta \in \Theta_0(\mathtt{r}_0)$:

$$\frac{1}{\sqrt{T}}\left\|\sum_t \mathbf{g}_t(\theta) - \lambda_t(\theta) - \mathbf{g}_t(\theta^*) + \lambda_t(\theta^*)\right\| \lesssim \delta_{T,\mathtt{x}} = \frac{(\mathtt{x} + \log T)^{3/4}}{T^{1/4}}. \tag{2.13}$$

With $\theta = \widetilde{\theta}$ and using $\sum_t \mathbf{g}_t(\widetilde{\theta}) = 0$, $\sum_t \lambda_t(\theta^*) = 0$ we get,

$$\left\|\sqrt{T}Q(\widetilde{\theta} - \theta^*) - \frac{1}{\sqrt{T}}\sum_t \mathbf{g}_t(\theta^*)\right\| \lesssim \delta_{T,\mathtt{x}}.$$

Similar to the proof of Theorem 2.3 in Spokoiny (2017), introducing the error of quadratic approximation of log-likelihood near the true parameter and provided (2.5) and (2.13), one can show that the square root of log-likelihood ratio is approximated with the same rate, i.e. $\left|\sqrt{2L(\theta) - 2L(\theta^*)} - \|\xi\|\right| \le \delta_{T,\mathtt{x}}$. Scaling $\mathtt{x} \leftarrow \mathtt{x} + \log 13$ provides the result.

2.7.4 Proof of Proposition 2.3

Similar to the original likelihood,

$$\zeta^\circ(\theta) = L^\circ(\theta) - \mathsf{E}^\circ L^\circ(\theta) = \sum_t (w_t - 1)\ell_t(\theta)$$

denotes the stochastic part of the likelihood in the bootstrap world.

Lemma 2.4. *Suppose 2.2, 2.3 and 2.6. For each* $\mathtt{x} \ge 1$ *with probability* $\ge 1 - 4e^{-\mathtt{x}}$ *w.r.t. to the data, the probability of*

$$\sup_{\theta \in \Theta(\mathtt{r})} \frac{1}{T^{1/2}}\left\|\sum_t (w_t - 1)\{\mathbf{g}_t(\theta) - \mathbf{g}_t(\theta^*)\}\right\| \le \diamondsuit^\flat(T, \mathtt{r}, \mathtt{x})$$

conditioned on the data is at least $1-3e^{-\mathtt{x}}$*, where*

$$\diamondsuit^{\flat}(T,\mathtt{r},\mathtt{x}) = C_3\left(\mathtt{r}\vee\sqrt{\mathtt{r}}+T^{-1/4}\{(\mathtt{rx})^{1/2}\vee(\mathtt{rx})^{1/4}\}+T^{-1/2}\mathtt{x}\right)\sqrt{\mathtt{x}+\log T},$$

with C_3 *not depending on* $T,\mathtt{r},\mathtt{x}$.

Proof. The proof is similar to that of Lemma 2.1. □

Corollary 2.2. *For* $\mathtt{x}\le\sqrt{T}$ *it holds with probability at least* $1-6e^{-\mathtt{x}}$*,*

$$\mathsf{P}^{\circ}\left(\sup_{\theta\in\Theta}\|\nabla\zeta^{\circ}(\theta)\|\le C_5T^{1/2}\sqrt{\mathtt{x}+\log T}\right)\le 1-5e^{-\mathtt{x}},$$

where C_5 *does not depend on* $T,\mathtt{x}$.

Now we are ready to state the global concentration result for the bootstrap estimator.

Proposition 2.4. *Assume 2.2-2.5 and 2.6. Then, on a set of probability at least* $1-12e^{-\mathtt{x}}$ *it holds with probability at least* $1-5e^{-\mathtt{x}}$ *conditioned on the data,*

$$\|\widetilde{\theta}^{\circ}-\theta^*\|\le C\sqrt{\frac{\mathtt{x}+\log T}{T}}.$$

Proof. Denote $r=\|\widetilde{\theta}^{\circ}-\theta\|$. Using Corollary 2.2 and the fact that $L^{\circ}(\widetilde{\theta}^{\circ})\ge L^{\circ}(\theta^*)$, we have on the event of probability at least $1-6e^{-\mathtt{x}}$ w.r.t. data, with probability at least $1-5e^{-\mathtt{x}}$ conditioned on the data, that

$$\begin{aligned} L(\widetilde{\theta})-L(\theta^*) \ge& L^{\circ}(\widetilde{\theta}^{\circ})-L^{\circ}(\theta^*)-\|\widetilde{\theta}^{\circ}-\theta^*\|\times\sup\|\nabla\zeta^{\circ}(\theta)\| \\ \ge& -C_5T^{1/2}r\sqrt{\mathtt{x}+\log T}. \end{aligned}$$

Using Proposition 2.1, we have that, additionally, on the other event of probability $1-6e^{-\mathtt{x}}$ it holds $r\lesssim\sqrt{r\sqrt{\frac{\mathtt{x}+\log T}{T}}}+\sqrt{\frac{\mathtt{x}+\log T}{T}}$, which yields the result. □

The rest can be accomplished using linear approximation of the score. Similar to the original likelihood, with $r_0=\|\widetilde{\theta}-\theta^*\|\vee\|\widetilde{\theta}^{\circ}-\theta^*\|$ it follows from (2.5),

$$\left\|\sum_t\lambda_t(\widetilde{\theta}^{\circ})-\sum_t\lambda_t(\widetilde{\theta})+TQ^2(\widetilde{\theta}^{\circ}-\widetilde{\theta})\right\|\le 2C_2Tr_0^2.$$

Here, $\sum_t \lambda_t(\theta)$ stands for the expectation of gradient of the likelihood. With help of Proposition 2.1 we first replace it with just the gradient, then, using Lemma 2.4 we replace it with the gradient of bootstrap likelihood. This finally leads to the proof of the proposition.

2.7.5 Proof of Theorem 2.1

W.l.o.g. we have an interval $\mathscr{I} = \{1, \ldots, T\}$ and a set of break points $\mathscr{S}(\mathscr{I}) \subset \mathscr{I}$ to be considered. Let us denote $\underline{T} = \alpha_0 T$ with $\alpha_0 > 0$ from the conditions of the theorem. We have by Proposition 2.2, that with probability at least $1 - e^{-\mathrm{x}}$ it holds for each $s \in \mathscr{S}(\mathscr{I})$,

$$\left| L_{A_{\mathscr{I},s}}(\widetilde{\theta}_{A_{\mathscr{I},s}}) - L_{A_{\mathscr{I},s}}(\theta^*) - \|\xi_{A_{\mathscr{I},s}}\|^2/2 \right| \leq \diamondsuit, \qquad \left| L_{B_{\mathscr{I},s}}(\widetilde{\theta}_{B_{\mathscr{I},s}}) - L_{B_{\mathscr{I},s}}(\theta^*) - \|\xi_{B_{\mathscr{I},s}}\|^2/2 \right| \leq \diamondsuit,$$
$$\left| L_{\mathscr{I}}(\widetilde{\theta}_{\mathscr{I}}) - L_{\mathscr{I}}(\theta^*) - \|\xi_{A_{\mathscr{I}}}\|^2/2 \right| \leq \diamondsuit,$$

where $\diamondsuit = C\underline{T}^{-1/4}(\mathrm{x} + \log T + \log(1 + 2|\mathscr{S}(\mathscr{I})|))^{3/4}$, implying

$$\left| L_{A_{\mathscr{I},s}}(\widetilde{\theta}_{A_{\mathscr{I},s}}) + L_{B_{\mathscr{I},s}}(\widetilde{\theta}_{B_{\mathscr{I},s}}) - L_{\mathscr{I}}(\widetilde{\theta}_{\mathscr{I}}) - (\|\xi_{A_{\mathscr{I},s}}\|^2 + \|\xi_{B_{\mathscr{I},s}}\|^2 - \|\xi_{\mathscr{I}}\|^2)/2 \right| \leq 3\diamondsuit.$$

By definition, $|\mathscr{I}|^{1/2}\xi_{\mathscr{I}} = |A_{\mathscr{I},s}|^{1/2}\xi_{A_{\mathscr{I},s}} + |B_{\mathscr{I},s}|^{1/2}\xi_{B_{\mathscr{I},s}}$, therefore for $\alpha = |A_{\mathscr{I},s}|/|\mathscr{I}|$ and $\beta = |B_{\mathscr{I},s}|/|\mathscr{I}| = 1 - \alpha$ we have,

$$\begin{aligned}
\|\xi_{A_{\mathscr{I},s}}\|^2 + \|\xi_{B_{\mathscr{I},s}}\|^2 - \|\xi_{\mathscr{I}}\|^2 =& \|\xi_{A_{\mathscr{I},s}}\|^2 + \|\xi_{B_{\mathscr{I},s}}\|^2 - \|\alpha^{1/2}\xi_{A_{\mathscr{I},s}} + \beta^{1/2}\xi_{B_{\mathscr{I},s}}\|^2 \\
=& \beta\|\xi_{A_{\mathscr{I},s}}\|^2 + \alpha\|\xi_{B_{\mathscr{I},s}}\|^2 - 2\alpha^{1/2}\beta^{1/2}\xi_{A_{\mathscr{I},s}}^{\top}\xi_{B_{\mathscr{I},s}} \\
=& \|\beta^{1/2}\xi_{A_{\mathscr{I},s}} - \alpha^{1/2}\xi_{B_{\mathscr{I},s}}\|^2
\end{aligned}$$

Obviously, similar expansion holds for the bootstrap counterpart, so that denoting

$$\begin{aligned}
S_{\mathscr{I},s} =& \frac{1}{\sqrt{|\mathscr{I}|}} \left[\sqrt{\frac{|B_{\mathscr{I},s}|}{|A_{\mathscr{I},s}|}} \sum_{t \in A_{\mathscr{I},s}} Q^{-1}\mathbf{g}_t(\theta^*) - \sqrt{\frac{|A_{\mathscr{I},s}|}{|B_{\mathscr{I},s}|}} \sum_{t \in B_{\mathscr{I},s}} Q^{-1}\mathbf{g}_t(\theta^*) \right], \\
S^{\circ}_{\mathscr{I},s} =& \frac{1}{\sqrt{|\mathscr{I}|}} \left[\sqrt{\frac{|B_{\mathscr{I},s}|}{|A_{\mathscr{I},s}|}} \sum_{t \in A_{\mathscr{I},s}} Q^{-1}w_t\mathbf{g}_t(\theta^*) - \sqrt{\frac{|A_{\mathscr{I},s}|}{|B_{\mathscr{I},s}|}} \sum_{t \in B_{\mathscr{I},s}} Q^{-1}w_t\mathbf{g}_t(\theta^*) \right],
\end{aligned}$$

we have

$$\left| \max_s T_{\mathscr{I},s} - \max_s \|S_{\mathscr{I},s}\|^2 \right| \leq 3\diamondsuit, \qquad \left| \max_s T^{\circ}_{\mathscr{I},s} - \max_s \|S^{\circ}_{\mathscr{I},s}\|^2 \right| \leq 3\diamondsuit. \tag{2.14}$$

For a single break point $s \in \mathscr{S}(\mathscr{I})$ by Azuma-Hoeffding inequality for all $\mathrm{x} > 0$ it holds,

$$\mathsf{P}\left(\|S_{\mathscr{I},s}\| \lesssim 1 + \sqrt{\mathrm{x}}\right) \geq 1 - e^{-\mathrm{x}},$$

so that it holds with probability $\geq 1 - e^{-\mathrm{x}}$,

$$\max_s \|S_{\mathscr{I},s}\| \lesssim \sqrt{\log T} + \sqrt{\mathrm{x}}, \qquad \max_s \|S^{\circ}_{\mathscr{I},s}\| \lesssim \sqrt{\log T} + \sqrt{\mathrm{x}}.$$

Additionally, for each $A \subset \mathscr{I}$ the covariance

$$\mathrm{Var}^{\circ}(\xi^{\circ}_A) = \frac{1}{|A|} \sum_{t \in A} Q^{-1} \mathbf{g}_t(\theta^*) \mathbf{g}_t(\theta^*)^{\top} Q^{-1}.$$

is concentrated near $\Sigma = \mathrm{Var}(Q^{-1}\mathbf{g}_1(\theta^*)) = Q^{-1}V^2Q^{-1}$, e.g. by Azuma-Hoeffding

$$\mathsf{P}\left(\|\mathrm{Var}^{\circ}(\xi^{\circ}_A) - \Sigma\| \lesssim \sqrt{\frac{1+\mathrm{x}}{|A|}}\right) \geq 1 - e^{-\mathrm{x}},$$

so that taking into account (2.7), it holds with probability $\geq 1 - e^{-\mathrm{x}}$, that for each $A = A_{\mathscr{I},s}$ or $A = B_{\mathscr{I},s}$ with $s \in \mathscr{S}(\mathscr{I})$,

$$\|\mathrm{Var}^{\circ}(\xi^{\circ}_A) - \Sigma\| \lesssim \sqrt{\frac{\log T + \mathrm{x}}{T}}. \tag{2.15}$$

Now we want to use Lemma A.4 with $n = T$. Since $\delta > 1$ by Assumption 2.4, we can choose $c_2, c' > 0$ such that $(1+\delta)/2 - (1+2\delta)c_2 > 1 + c'$. Then, we can have $a, \varepsilon > 0$ such that $a + \varepsilon < \frac{1}{2} - 2c_2$ and $c_2 + (1+\delta)a > 1 + c'$. Setting $b = a + \gamma + \varepsilon$, we have that

$$1 - b - \gamma a < -c', \qquad b < \frac{1}{2} - c_2, \qquad b - a > c_2.$$

This means, that taking $q = \lceil T^a \rceil$ and $r = \lceil T^b \rceil$ and $D_n \lesssim \sqrt{\log n}$ by Assumption 2.6, the conditions of Lemma A.4 are satisfied. Moreover, by (2.15) we have $\Delta \lesssim \sqrt{\log T / T}$ with probability $\geq 1 - 1/(2T)$, so that for each $t, y \in \mathbb{R}$

$$\left|\mathsf{P}(\max_s \|S_{\mathscr{I},s}\| > t) - \mathsf{P}(\max_s \|S^{\circ}_{\mathscr{I},s}\| > t + y)\right| \lesssim T^{-c \wedge c'} + |y| \log^{1/2} T. \tag{2.16}$$

Thus, for $|y| \le 6\diamondsuit$ taken for $\mathtt{x} = C\log T$, we have for each $t, y \in \mathbb{R}$

$$\sup_t \left| \mathsf{P}(\max_s T_{\mathscr{I},s} > t + y) - \mathsf{P}(\max_s T^{\circ}_{\mathscr{I},s} > t) \right| \lesssim T^{-c\wedge c'} + |y| \log^{1/2} T$$

with probability $\ge 1 - 1/T$.

2.7.6 Proof of Lemma 2.3

Note, that integrating the inequality (2.5) with $Q = \sum_{i=1}^n \mathsf{E} f_{it}(0) \nabla q_{it}(\theta^*)[\nabla q_{it}(\theta^*)]^\top$, we get second-order approximation in the neighbourhood of θ^*,

$$\left| \frac{1}{T}\mathsf{E}L(\theta) - \frac{1}{T}\mathsf{E}L(\theta^*) + \|Q(\theta - \theta^*)\|^2/2 \right| \le C\|\theta - \theta^*\|^3,$$

therefore we get that for $\|\theta - \theta^*\| > \mathtt{r}$ and $\mathtt{r} \le \mathtt{r}_0 = \lambda_{\min}(Q^2)/(4C)$ we have

$$\frac{1}{T}\mathsf{E}L(\theta) - \frac{1}{T}\mathsf{E}L(\theta^*) < -b_{loc}\mathtt{r}^2, \qquad b_{loc} = \lambda_{\min}(Q^2)/4.$$

Next, notice that if a r.v. Z has τ quantile 0, then for $\delta > 0$

$$\begin{aligned}
\mathsf{E}\rho_\tau(Z+\delta) - \mathsf{E}\rho_\tau(Z) =& \mathsf{E}(Z+\delta)(\tau - \mathbf{I}[Z+\delta \le 0]) - \mathsf{E}Z(\tau - \mathbf{I}[Z \le 0]) \\
=& \delta\mathsf{E}(\tau - \mathbf{I}(Z \le \delta) + \mathbf{I}[Z \in (-\delta, 0)]) + \mathsf{E}Z\mathbf{I}(Z \in (-\delta, 0)) \\
=& \mathsf{E}(Z+\delta)\mathbf{I}(Z \in (-\delta; 0)) \\
\ge& \delta/2\mathsf{E}\mathbf{I}(Z \in (-\delta/2; 0)) \\
\ge& \frac{\underline{f}\delta}{2}\left(\frac{\delta}{2} \wedge \delta_0\right),
\end{aligned}$$

and by analogy same bound takes place for $\mathsf{E}\rho_\tau(Z-\delta) - \mathsf{E}\rho_\tau(Z)$. Therefore,

$$\mathsf{E}\ell_t(\theta) - \mathsf{E}\ell_t(\theta^*) \le \mathsf{E}\sum_{i=1}^n \frac{\underline{f}|q_{it} - q_{it}^*|}{2}\left(\frac{|q_{it} - q_{it}^*|}{2} \wedge \delta_0\right),$$

where due to (2.4), the right-hand side is bounded by $\underline{f}\delta(\delta \wedge \delta_0)/4$ with $\delta = \delta(\mathtt{r}_0)$. Setting $b_{glob} = \underline{f}\delta(\delta \wedge \delta_0)/(4\mathtt{r}_0^2)$, we get that the required inequality is satisfied with $b = b_{loc} \wedge b_{glob}$.

2.7.7 Proof of Corollary 2.1

Let $z(\alpha)$ denotes $(1-\alpha)$-quantile of the test T, and $z^{\circ}(\alpha)$ is that of T° with respect to the bootstrap probability (here for convenience we write the confidence level in the brackets). Since $\mathsf{P}(X+Y > a+b) \leq \mathsf{P}(X > a) + \mathsf{P}(Y \geq b)$ for arbitrary random variables X, Y and real numbers a, b, we have for each $\delta \in (0;\alpha)$

$$\begin{aligned} \mathsf{P}(T > z^{\circ}(\alpha)) \leq & \mathsf{P}(T > z(\alpha+\delta)) + \mathsf{P}(z^{\circ}(\alpha) \leq z(\alpha+\delta)) \\ = & \alpha + \delta + \mathsf{P}(z^{\circ}(\alpha) \leq z(\alpha+\delta)), \\ \mathsf{P}(T > z^{\circ}(\alpha)) \geq & \mathsf{P}(T > z(\alpha-\delta)) - \mathsf{P}(z^{\circ}(\alpha) \geq z(\alpha-\delta)) \\ = & \alpha - \delta - \mathsf{P}(z^{\circ}(\alpha) \geq z(\alpha-\delta)). \end{aligned} \tag{2.17}$$

Furthermore,

$$\begin{aligned} \mathsf{P}(z^{\circ}(\alpha) \geq z(\alpha-\delta)) = & \mathsf{P}\left\{\mathsf{P}^{\circ}(T^{\circ} > z(\alpha-\delta)) \geq \alpha\right\}, \\ \mathsf{P}(z^{\circ}(\alpha) \leq z(\alpha+\delta)) = & \mathsf{P}\left\{\mathsf{P}^{\circ}(T^{\circ} > z(\alpha+\delta)) \leq \alpha\right\}. \end{aligned}$$

By Theorem 2.1 we have on a set of probability $\geq 1 - 1/T$, that

$$\sup_t |P(T > t) - P^{\circ}(T^{\circ} > t)| \leq CT^{-c}.$$

Taking $\delta = 2CT^{-c}$ and $t = z(\alpha-\delta)$ we have,

$$P^{\circ}(T^{\circ} > z(\alpha-\delta)) \leq \alpha - \delta + CT^{-c} < \alpha$$

and in a similar way,

$$P^{\circ}(T^{\circ} > z(\alpha+\delta)) \geq \alpha + \delta - CT^{-c} > \alpha.$$

Thus, with this choice of δ it holds,

$$\mathsf{P}(z^{\circ}(\alpha) \leq z(\alpha+\delta)) \leq 1/T, \qquad \mathsf{P}(z^{\circ}(\alpha) \geq z(\alpha-\delta)) \leq 1/T,$$

which via (2.17) concludes the proof.

Chapter 3

Influencers and Communities in Social Networks

Financial and social networks are often analysed through *vector autoregression* model, for instance, in Härdle et al. (2019). Consider a network that produces a time series $Y_t \in \mathbb{R}^N$, $t = 1, \dots, T$ and dependencies between it's elements are modeled through the equation

$$Y_t = \Theta Y_{t-1} + W_t, \tag{3.1}$$

where W_t are innovations that satisfy $\mathsf{E}[W_t | \mathscr{F}_{t-1}] = 0$, $\mathscr{F}_t = \sigma\{Y_{t-1}, Y_{t-2}, \dots\}$, so that the interactions between the nodes are described by an autoregression operator $\Theta \in \mathbb{R}^{N \times N}$. In terms of the network connections we say that a node i is connected tothe node j if

$$\Theta_{ij} \neq 0,$$

so that the adjacency matrix of such network is represented by nonzero coefficients and the sparsity of Θ represents number of the edges. For large-scale time series one encounters the curse of dimension, as estimating the matrix-parameter Θ with N^2 elements requires significantly large number of observations T.

Several attempts to reduce the dimensionality have been made in the past literature. Assuming that the elements of a time series form a connected network, Zhu et al. (2017) introduces a Network Autoregression model (NAR) with $\Theta_{ij} = \beta A_{ij} / \sum_{k=1}^{N} A_{ik}$, provided that the adjacency matrix $A \in \mathbb{R}^{N \times N}$ is known. Here, the regression operator, defined up to a single parameter β, which called a *network effect*, can be estimated through a simple least squares. Zhu et al. (2016) also extend this model for conditional quantiles. Furthermore, Zhu and Pan (2017) argue that a single network parameter may not be satisfactory as it treats all nodes of the network homogeneously. In particular, the NAR model implies that each node is affected by it's neighbours in the same extent, while in reality we may have financial institutions that are affected less than the others, thus more secure and risk-free. They then propose to detect communities in the network based on the given adjacency matrix and suggest that the nodes in each community share a separate network effect parameter. A somewhat opposite direction is taken by Gudmundsson and Brownlees (2018): their BlockBuster algorithm determines the communities through the estimated autoregressive model, which, however, does not solve the dimensionality problem. Apart from this line of work, sparse regularisations have been extensively used, see Fan et al. (2009); Han et al. (2015); Melnyk and Banerjee (2016).

To sum up we want to address the following problems, which one encounters dealing with vector autoregression:

- as already mentioned above, in VAR the parameter dimension is particularly large and requires even larger time intervals for consistent estimation. Even if one can afford such data set, in the long run, autoregressive parametric models tend to be violated, see e.g. Čížek et al. (2009). Naturally, we want to impose some structural assumptions on the operator Θ, so that it can be estimated by means of moderate sample sizes.

- The NAR model assumes that the adjacency matrix is given. In particular, this is justified for social networks with a natural friendship/follower-followee relationship. For a network of financial institutions, there is no explicitly defined adjacency matrix and one has to heuristically evaluate it using additional information (identical shareholders, trading volumes, etc.) or through analysing correlations and lagged cross-correlations between returns or risk profile, see Diebold and Yılmaz (2014) and Chen et al. (2019b). However, there is no rigorous reason to believe that the operator in (3.1) depends explicitly on such adjacency matrix, see also Cha et al. (2010).

Motivated by two aspects of social networks we construct a new *Social Network autoregression with Influencers and Communities* model (SoNIC). Based on a user experience on platforms like facebook, twitter, etc., one can assume that there are some users that are followed significantly more than the others. Take, for example, celebrities, sportsmen, politicians, or instagram divas. These nodes of a network have much more influence over the others, than the rest of the nodes. We call such nodes *influencers*. In the notation of autoregressive parameter, a node j is called an influencer, if there is a significant amount of other nodes i such that $\Theta_{ij} \neq 0$. Assuming that the number of influencers is limited, we can say that only few columns of matrix Θ are important. This allows us to take into account only the connections to the influencers, significantly reducing the number of parameters to be estimated. A similar idea is used in Chen et al. (2018), with a group-lasso regularisation imposed, so that they find a solution with few active columns. Notice, however, that only relying on sparsity still requires $T > N$, see e.g. Chernozhukov et al. (2018); Fan et al. (2009).

It is also widely known that social networks consist of smaller communities, with the nodes exhibiting higher connection density or similar behaviour inside the communities. Zhu and Pan (2017) makes one step to extend the NAR model from Zhu et al. (2017) into a more realistic set-up by saying that instead of a single network effect parameter, there are separate parameters for each community. For us the behaviour of a node i is characterized by the coefficients $\Theta_{i1}, \dots, \Theta_{iN}$, i.e. the nodes it depends on and to what extent. We assume that the nodes are separated into few clusters such that the nodes from the same cluster have the same dependencies. This brings a bigger picture into the view: instead of saying that two nodes from the same cluster are more likely to be connected, we say that they are connected to the same influencers.

Our main focus is application to sentiment extracted from a microblogging platform dedicated to stock trading, StockTwits[1]. For each user one can extract average sentiment weight over the messages he posts during the day. Analysing the resulting time series we are able to identify, on one hand, influencers — the users whose opinion is most important, and on the other, different communities. Another problem that we want to address is the presence of missing observation in the data set, since on some days some users do not leave any messages. We treat this as follows: the there is an underlying opinion process that follows autoregressive equation (3.1), while the users decide whether to express it or not during each day.

The rest of the chapter is organized as follows. Section 3.1 introduces the reader to StockTwits platform, describes in detail the available data set and the process of sentiment weights extraction. In Section 3.2 first introduces our SoNIC model, then describes the estimation procedure and provides a consistency result. In Section 3.3 we provide simulation results that partially confirm the theoretical properties of our estimator. Next, in Section 3.4 we present and discuss the results of application of our model to some data sets extracted

[1]https://stocktiwts.com

from the StockTwits. Section 3.5, as well as Sections 3.6, A.1 in the appendix, are dedicated to the proofs.

3.1 StockTwits

Among social media platforms, we particularly are interested in StockTwits[2] for a number of reasons. Firstly, it becomes predominantly popular and stands for a leading social network for investors and traders. Secondly, it is similar to Twitter, but dedicated to financial discussion. One of features leads to its popularity is a well-designed reference between the message content and the referring stock symbols. Conversations are organized around 'cashtags' (e.g. '$AAPL' for APPLE; '$BTC.X' for BITCOIN) that allow to narrow streams down to specific assets. Thirdly and most importantly, users can also express their sentiment/opinions by labeling their messages as 'Bearish' (negative) or 'Bullish' (positive) *via* a toggle button. These are so-called *self-report sentiment*. Indeed, the user generated messages and self-reported sentiment attract the researchers for sentiment analysis. The available labeled data benefits an advance on textual analysis that typically relies on the available training dataset. We use this convention and StockTwits Application Programming Interface (API) to download all messages containing the preferred cashtages. StockTwits API also provides for each message its user's unique identifier, the time it was posted at with a one-second precision, and the sentiment associated by the user ('Bullish', 'Bearish' or unclassified).

Among over thousand tickers/symbols, we particularly pick up two selective symbols, $AAPL for APPLE; $BTC.X for BITCOIN, which represents the most popular security and cryptocurrency, respectively. We conjecture that due to the fact they attract investors/users with very distinct risk preference, the resulting opinion networks and its dynamics may exhibit diverse structures. In Table 3.1 we summarize the messages' statistics with respect to AAPL and bitcoin. Even though we exclusively consider these two symbols, the message volume and number of users associated with these two symbols are tremendous. A glimpse of table shows different profiles between two symbols. Firstly, the users who interest in BTC tend to disclose their sentiment, evident by 44% of labelled messages, while in AAPL only 28% of messages are labelled. It may lead to a better training accuracy in the case of BTC messages relative to the training model based on AAPL. Secondly, there is a clear imbalance between the numbers of positive and negative messages, showing that online investors are optimistic on average, as previously found by Kim and Kim (2014) or Avery et al. (2016). It

[2]https://stocktwits.com/

seems that the imbalance is more evident in the case of AAPL. Through the reported average message volume per day, there is no doubt that AAPL is more able to attract attentions from potential investors than BTC could.

Symbols	AAPL	BTC
message volume	449,761	644,597
number of distinct users	26,521	25,492
number of bullish messages	133,316	196,555
number of bearish messages	48,186	90,677
percentage of bullish messages	20.6%	30.4%
percentage of bearish messages	7.4%	14.0%
percentage of labeled messages	28.0%	44.4%
size of positive training dataset	99,985	147,759
size of negative training dataset	36,100	67,752
message volume per day	730	305
number of positive terms in lexicon	4,000	3,775
number of negative terms in lexicon	4,000	3,759
sample period	2017-05-22	2013-03-21
	2019-01-27	2018-12-27

Table 3.1 Summary statistics of social media messages

3.1.1 Quantifying message content

In order to study the sentiment interaction of users and the dynamics of interaction, one needs to quantify the messages from the selected users and subsequently model the quantified texts from ultra high-dimensional users. In practice, converting text data into a quantitative sentiment variable can been done by two techniques, namely dictionary-based and machine learning-based analysis. Although a machine learning technique has many advantages compared to a dictionary-based approach, a dictionary-based approach offers better transparency, explication and less computational burden. Loughran and McDonald (2016) recommend that alternative complex methods (machine learning) should be considered only when they add substantive value beyond simpler and more transparent approaches such as bag-of word. We therefore opt for the lexicon approach in the task of sentiment quantification.

A dictionary, or lexicon, is a list of words labeled as positive, negative or neutral. Assuming such a list, the classic *bag-of-words* approach consists of counting the number of positive and negative words in a document in order to assign it a sentiment value or tone. For example, a simple dictionary containing only the words 'good' and 'bad' with respectively positive and negative labels would classify the sentence 'Bitcoin is a good investment' as positive, with a tone of +1. As known by literature, the simplicity of the dictionary-based approach guarantees transparency and replicability provided, on the cons side, it comes with limitations associated with natural language analysis. First, referring in Deng et al. (2017) to the 'context of a discourse', one needs to be aware of the content domain, to which language interpretation is sensitive. For example, Loughran and McDonald (2011) point that words like 'tax' or 'cost' are classified as negative by Harvard General Inquirer lexicon, whereas they should be considered neutral in financial context. Another example is about quantifying sentiment toward cryptocurrency, playing as non-standard assets and embracing new technologies as part of asset characteristics. Chen et al. (2019a) point out that in many domain-specific terms, such as blockchain, ICO, hackers, wallet, shitcoin and binance, 'hodl', are not covered in existing financial or psychological dictionaries. They create a novel cryptocurrency lexicon in response to the need of adopting a specific approach to measure sentiment about cryptocurrencies. The second limitation is the one of language domain defined by Deng et al. (2017) as the 'lexical and syntactical choices of language'. One example would be the difference between newspapers where a formal and standardized tone is mostly used, and social media, where slang and emojis are preponderant (Loughran and McDonald, 2016). As shown by Chen et al. (2019a), online investors also use new 'emojis' such as 🚀 (positive) and 💩 (negative) when talking about cryptocurrencies, which are obviously also not collected in traditional dictionary.

To balance the complexity and transparency and also take into account the domain-specific terms in social media while applying lexicon approach, in the sentiment quantification for the messages of AAPL we employ the social media lexicon developed by Renault (2017a) while in the quantification of BTC messages we advocate the lexicon tailored for cryptocurrency asset by Chen et al. (2019a). Renault (2017a) demonstrates that his constructed lexicon significantly outperforms the benchmark dictionaries (Loughran and McDonald, 2016) used in the literature while remaining competitive with more complex machine learning algorithms. On the basis of 125,000 bullish and another 125,000 bearish messages published on StockTwits, using the lexicon for social media achieves 90% of classified messages, and 75.24% of correct classifications. With a collection of 1,533,975 messages from 38,812 distinct users, posted between March 2013 and December 2018, and related to 465 cryptocurrencies listed in StockTwits [3], Chen et al. (2019a) documents that

[3]This list can be found at https://api.stocktwits.com/symbol-sync/symbols.csv

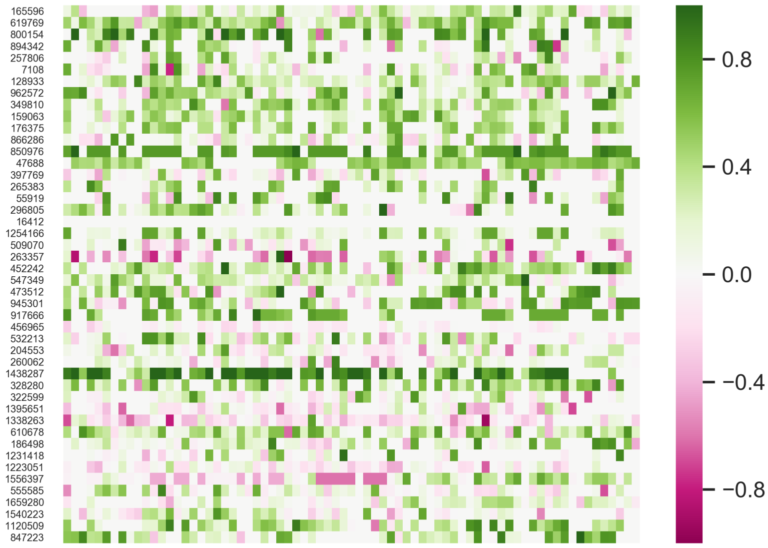
implementing the crypto lexicon is able to classify 83% of messages, with 86% of them being correctly classified.[4]

The natural language processing (NLP) is prerequisite while implementing textual analysis. Following by Sprenger et al. (2014) and Renault (2017b), we convert unstructured text into clean and manageable textual content as the grounding base throughout the textual analysis. First, all messages are lowercased. To account for lengthening of words, which has been shown to be a critical feature of sentiment expression on microblogs (Brody and Diakopoulos, 2011), but avoid noise in the lexicon, sequences of repeated letters are shrink to a maximum length of 3. Tickers ('$BTC.X', '$LTC.X'...), dollar or euro values, hyperlinks, numbers and mentions of users are respectively replaced by the words 'cashtag', 'moneytag', 'linktag', 'numbertag' and 'usertag'. The prefix "negtag_" is added to any word consecutive to 'not', 'no', 'none', 'neither', 'never' or 'nobody'. Finally, the three stopwords 'the', 'a', 'an' and all punctuation except the characters '?' and '!' are removed. Exclamation and interrogation marks are kept as it has been previously shown that they are often part of significant bigrams that improve lexicon accuracy (Renault, 2017b).

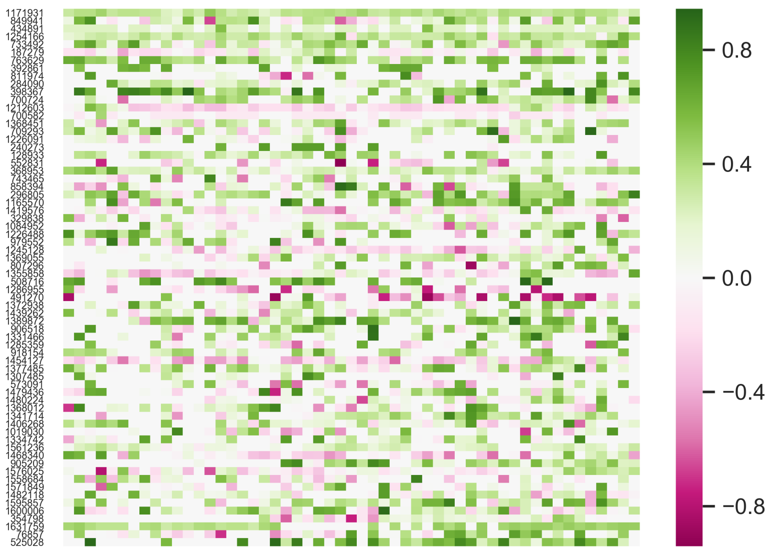
The next step is to undertake the lexicon approach in order to extract the semantic expression, sentiment or opinions. For each individual message in Table 3.1, we filter the terms being collected in the designated lexicon, and equally weight the filtered terms as the message sentiment score. Since the designated lexicon are weighted lexicon and in the range of -1 and $+1$, the sentiment score is automatically in the same range.

To visualize the quantified sentiment from individuals over time, we select the most active users and display their daily sentiment from 2018-11-01 to 2018-12-27. The heatmap shown in Figure 3.1 is a 2-dimensional matrix with y-axis for user's ID and x-axis for message posting date, the cell of heatmap is the quantified sentiment whose magnitude is represented as the color coded in the adjunct color bar. The evolution and dynamics of sentiment among users can be read in such heatmap presentation. From either Figure 3.1a (AAPL) or Figure 3.1b (BTC), one observes the similar color codes among a subset of users at particular date or period, indicating a contemporaneous common opinion/sentiment and an intertemporal opinion flow among users. Worth noting that some heterogeneity may exist as some users possess optimistic opinions and others are persistently pessimistic.

[4]The percentage of of correct classification is defined as the proportion of correct classifications among all classified messages, while the percentage of classified messages is denoted as the proportion of classified messages among all messages. See more detain in Renault (2017a) and Chen et al. (2019a)



(a) AAPL users



(b) BTC users

Figure 3.1 Social media users' sentiment over time
y-axis is the user's id, while x-axis is time stamp from 2018-11-01 —a 2018-12-27.

3.2 Main results

3.2.1 Clusters of nodes and influencers

In our set-up the behaviour of each node $i \in [N]$ is characterized by the coefficients $\Theta_{i1}, \dots, \Theta_{iN}$, and when we group the nodes using their characteristics the notion of community is merged with the notion of cluster. We assume that the nodes are separated into clusters, such that these coefficients remain the same for the nodes within each cluster. Let us first give a precise definition of a clustering.

Definition 3.1. *A K-*clustering *of the set of the nodes $[N]$ is called a sequence $\mathscr{C} = (C_1, \dots, C_K)$ of K subsets of $[N]$, such that*

- *any two subsets are disjoint $C_i \cap C_j = \emptyset$ for $i \neq j$;*
- *the union of subsets C_j gives all nodes,*

$$C_1 \cup \dots \cup C_K = \{1, \dots, N\}.$$

Two clusterings $\mathscr{C}$ and $\mathscr{C}'$ are equivalent, if there is a permutation π on $\{1, \dots, K\}$, such that the clusters are equal with respect to relabelling, i.e. $C_j = C'_{\pi(j)}$ for each $j = 1, \dots, K$.

Furthermore, denote a distance between two clusterings is defined as

$$d(\mathscr{C}, \mathscr{C}') = \min_{\pi} \sum_{j=1}^{K} |C_j \setminus C'_{\pi(j)}|.$$

Remark 3.1. *The distance between clusterings is in fact the minimal amount of node transfers from one cluster to another, that is required to make the clusterings equivalent. To see this, notice that each clustering can be defined as a sequence $(l_1, \dots, l_N)$ of N labels taking values in $\{1, \dots, K\}$, so that each cluster defines as $C_j = \{i : \ l_i = j\}$. Then, if the clustering $\mathscr{C}'$ corresponds to the labels $l'_1, \dots, l'_N$, it is not hard to see, that the distance between them equals to*

$$d(\mathscr{C}, \mathscr{C}') = \min_{\pi} \sum_{i=1}^{N} \mathbf{I}(l_i \neq \pi(l'_i)).$$

We specify our model by putting structural assumptions which are motivated by both the communities and presence of the influencers.

Definition 3.2. *We say that* $\Theta \in \mathsf{SoNIC}(s,K)$ *(Social Network with Influencers and Communities) if*

- *each user is influenced by at most s influencers, i.e.*

$$\max_i \sum_{j=1}^{N} \mathbf{I}(\Theta_{ij} \neq 0) \leq s;$$

- *there is a K-clustering* $\mathscr{C} = (C_1, \ldots, C_K)$ *such that*

$$\Theta_{ij} = \Theta_{i'j}, \qquad j = 1, \ldots, N$$

whenever i, i' *are from the same cluster* C_l, $l = 1, \ldots, K$.

We will also say that Θ *has clustering* $\mathscr{C}$.

Once $\Theta \in \mathsf{SoNIC}(s,K)$ has clustering $\mathscr{C} = (C_1, \ldots, C_K)$, the following factor representation takes place

$$\Theta = Z_{\mathscr{C}} V^{\top}, \tag{3.2}$$

where $Z_{\mathscr{C}}, V$ are $N \times K$ matrices such that

- $Z_{\mathscr{C}} = [\mathbf{z}_{C_1}, \ldots, \mathbf{z}_{C_K}]$ is a normalized index matrix of clustering $\mathscr{C}$, where for any $C \subset [N]$ we denote

$$\mathbf{z}_C = \frac{1}{\sqrt{|C|}} (\mathbf{I}(1 \in C), \ldots, \mathbf{I}(N \in C)) \in \mathbb{R}^N$$

— a normalized index vector for the cluster C;

- $V = [\mathbf{v}_1, \ldots, \mathbf{v}_K]$ has sparse columns,

$$\|\mathbf{v}_j\|_0 \leq s.$$

A schematic picture of what we expect is shown in Figure 3.2. Here, the nodes from the same clusters depend on the same influencers (the grey nodes may be in any of the clusters), which also coincides with the idea of Rohe et al. (2016), who look for the right-hand side singular vectors of the Lagrangian in a directed network, grouping the nodes who tend to be affected by the same group of nodes.

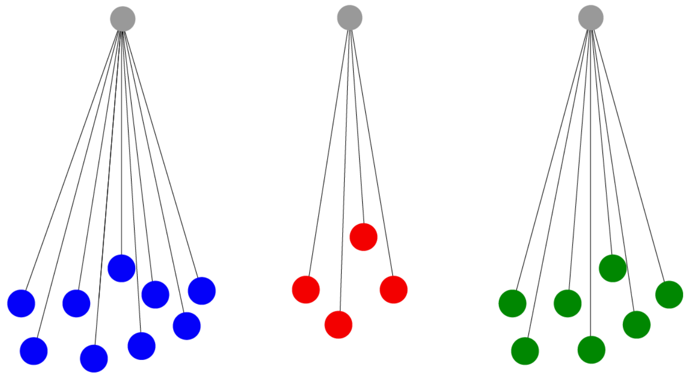


Figure 3.2 Example of a network with influencers.

The equation (3.2) is akin to bilinear factor models, which appear in Econometric models with factor loadings, see e.g. Moon and Weidner (2018) and the references therein. It is also a popular machine learning technique for low rank approximation, see a thorough review in Udell et al. (2016). Chen and Schienle (2019) use sparse factors for a closely related model.

3.2.2 Model with missing observations

A network of size N represents a multivariate time series $Y_t = (Y_{1t}, \dots, Y_{Nt})^\top \in \mathbb{R}^N$, where Y_{it} is the response of a node $i = 1, \dots, N$ at a time $t = 1, \dots, T$, that follows the autoregressive equation

$$Y_t = \Theta^* Y_{t-1} + W_t,$$

with $\mathsf{E}[W_t | \mathscr{F}_{t-1}] = 0$ for $\mathscr{F}_{t-1} = \sigma(W_{t-1}, W_{t-2}, \dots)$. Once $|||\Theta^*|||_{\mathsf{op}} < 1$ the process exists as a converging series

$$Y_t = \sum_{k \geq 0} (\Theta^*)^k W_{t-k}, \tag{3.3}$$

and if the covariance of the innovations is $S = \mathrm{Var}(W_t)$, then the covariance of the process reads as

$$\Sigma = \mathrm{Var}(Y_t) = \sum_{k \geq 0} (\Theta^*)^k S \{(\Theta^*)^k\}^\top.$$

For simplicity we consider *subgaussian* vectors W_t, as it allows to have deviation bounds for covariance estimation with exponential probabilities. Recall the following definition, see e.g. Vershynin (2018).

Definition 3.3. *A random vector $W \in \mathbb{R}^d$ is called L-subgaussian if for arbitrary $\mathbf{u} \in \mathbb{R}^d$ it holds*

$$\|\mathbf{u}^\top W\|_{\psi_2} \le L\|\mathbf{u}^\top X\|_{L_2},$$

where for a random variable $X \in \mathbb{R}$ we denote

$$\|X\|_{\psi_2} = \inf\{C > 0 : \, \mathsf{E}e^{\left(\frac{|X|}{C}\right)^2} \le 2\},$$
$$\|X\|_{L_2} = \mathsf{E}^{1/2}|X|^2.$$

Additionally, we adopt the framework of Lounici (2014) for vectors with missing observations, assuming that each variable Y_{it} is either observed or not independently and with some probability. Formally speaking, instead of having a realisation of the whole vector Y_t we only have access to the vectors of form

$$Z_t = (\delta_{1t}Y_{1t}, \dots, \delta_{Nt}Y_{Nt})^\top, \qquad t = 1, \dots, T, \tag{3.4}$$

where $\delta_{it} \sim \mathrm{Be}(p_i)$ are independent Bernoulli random variables for each $i = 1, \dots, N$ and $t = 1, \dots, T$ and some $p_i \in (0, 1]$. This means that each variable Y_{it} is only observed with probability p_i independently from the other variables, with $\delta_{it} = 1$ corresponding to observed Y_{it} and $\delta_{it} = 0$ to missing Y_{it}, so instead we simply receive zero. Obviously, the case $p_i = 1$ for each $i = 1, \dots, N$ corresponds to the process without missing observations, therefore the new problem serves as a generalisation and the results for the missing observations model can be applied in the regular case as well.

Remark 3.2. *In terms of the StockTwits sentiment we interpret the process Y_t as unobserved underlying* opinion process. *During each day the users decide whether to express their opinion or not by posting a message on their page, which results in a masked process Z_t. Since some users are more active than the others, we need to account for different probabilities p_i.*

Suppose, that the probabilities p_i are given (otherwise they can easily be estimated) and set $\mathbf{p} = (p_1, \dots, p_N)^\top$. Due to Lounici (2014), set the observed empirical covariance $\Sigma^* = \frac{1}{T}\sum_{t=1}^T Z_t Z_t^\top$ and consider the following covariance estimator,

$$\hat{\Sigma} = \mathrm{diag}\{\mathbf{p}\}^{-1}\,\mathrm{Diag}(\Sigma^*) + \mathrm{diag}\{\mathbf{p}\}^{-1}\,\mathrm{Off}(\Sigma^*)\,\mathrm{diag}\{\mathbf{p}\}^{-1}.$$

It is straightforward to calculate that this is an unbiased estimator, i.e.

$$\mathsf{E}\hat{\Sigma} = \Sigma.$$

The following lemma provides deviation bounds restricted to a subspace of a dimension lower than the process itself.

Theorem 3.1. *Assume the vectors W_t are independent L-subgaussian and also*

$$|||\Theta|||_{\mathsf{op}} \leq \gamma < 1, \qquad p_i \geq p_{\min} > 0.$$

Let $P, Q \in \mathbb{R}^{N\times N}$ be two arbitrary orthogonal projectors of rank M_1, M_2, respectively. Then, for any $u \geq 1$it holds with probability at least $1 - e^{-u}$,

$$|||P(\hat{\Sigma} - \Sigma)P|||_{\mathsf{op}} \leq C|||S|||_{\mathsf{op}} \left(\sqrt{\frac{M_1 \vee M_2 (\log N + u)}{T p_{\min}^2}} \bigvee \frac{\sqrt{M_1 M_2}(\log N + u)\log T}{T p_{\min}^2} \right),$$

where $C = C(\gamma, L)$ only depends on L and γ.

See proof of this result in Section 3.6.

Additionally, we are interested in estimating lag-1 cross-covariance under the same scenario. Namely, based on the sample $Z_1, \ldots, Z_T$ and given the probabilities $p_1, \ldots, p_N$ we wish to estimate the matrix $A = \mathsf{E} Y_t Y_{t+1}^\top$. Since $\mathsf{E}[Y_{t+1} | \mathscr{F}_t] = \Theta Y_t$ for the linear process (3.19), the corresponding cross-covariance reads as

$$A = \Sigma\Theta.$$

Consider the following estimator

$$\hat{A} = \operatorname{diag}\{\mathbf{p}\}^{-1} A_T^* \operatorname{diag}\{\mathbf{p}\}^{-1},$$

where A^* is the observed empirical cross-covariance

$$A^* = \frac{1}{T-1} \sum_{t=1}^{T-1} Z_t Z_{t+1}^\top.$$

For this estimator we provide an upper-bound, again with a restriction to some low-dimensional subspaces.

Theorem 3.2. *Let P, Q be two projectors of rank M_1 and M_2, respectively. Assume the vectors W_t independent are L-subgaussian and also*

$$\|\Theta\|_{\mathsf{op}} \le \gamma < 1, \qquad p_i \ge p_{\min} > 0.$$

Then, for any $u \ge 1$ it holds with probability at least $1 - e^{-u}$

$$\|P(\hat{A} - A)Q\|_{\mathsf{op}} \le C\|S\|_{\mathsf{op}} \left(\sqrt{\frac{(M_1 \vee M_2)(\log N + u)}{T p_{\min}^2}} \bigvee \frac{\sqrt{M_1 M_2}(\log N + u)\log T}{T p_{\min}^2} \right),$$

where $C = C(\gamma, L)$ only depends on γ and L.

The proof is postponed to Section 3.6.

3.2.3 Alternating minimization algorithm

In order to estimate the matrix $\Theta = Z_{\mathscr{C}} V^\top$ we need to estimate both $\mathscr{C}$ and V simultaneously. Suppose, we have some clustering $\mathscr{C}$ at hand and we want to estimate the corresponding V. The mean squared loss from the fully observed sample would like as follows,

$$\begin{aligned} R^*_{\mathscr{C}}(V) =& \frac{1}{2(T-1)} \sum_{t=1}^{T-1} \|Y_{t+1} - Z_{\mathscr{C}} V^\top Y_t\|^2 \\ =& \frac{1}{2}\operatorname{tr}(V^\top \tilde{\Sigma} V) - \operatorname{tr}(V^\top \tilde{A} Z_{\mathscr{C}}) + \frac{1}{2(T-1)} \sum_{t=1}^{T-1} \|Y_{t+1}\|^2, \end{aligned}$$

where we used the fact that $Z_{\mathscr{C}}^\top Z_{\mathscr{C}} = I_K$ and the trace of matrix product is invariant with respect to transition $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Here, we also denote

$$\tilde{\Sigma} = \frac{1}{T-1} \sum_{t=1}^{T-1} Y_t Y_t^\top, \qquad \tilde{A} = \frac{1}{T-1} \sum_{t=1}^{T-1} Y_t Y_{t+1}^\top,$$

to be empirical covariance and empirical lag-1 covariance built on a sample $Y_1, \ldots, Y_T$, which we do not fully observe. Instead, since we only have access to the missing observation estimators $\hat{\Sigma}$ and $\hat{A}$, consider the loss function (notice that the star has disappeared)

$$R_{\mathscr{C}}(V) = \frac{1}{2}\operatorname{tr}(V^\top \hat{\Sigma} V) - \operatorname{tr}(V^\top \hat{A} Z_{\mathscr{C}}).$$

As we are searching for a sparse matrix V, we additionally put a lasso regularization, so we end up with the following program,

$$\hat{V}_{\mathscr{C},\lambda} = \arg\min R_{\mathscr{C},\lambda}(V), \qquad R_{\mathscr{C},\lambda}(V) = R_{\mathscr{C}}(V) + \lambda\|V\|_{1,1}$$
$$= \frac{1}{2}\operatorname{tr}(V^\top \hat{\Sigma} V) - \operatorname{tr}(V^\top \hat{A} Z_{\mathscr{C}}) + \lambda\|V\|_{1,1},$$

where $\|V\|_{1,1} = \sum_{ij}|V_{ij}|$, and $\lambda > 0$ somehow depends on the dimension N and number of observations T. Concerning this minimization problem we have the following observations:

- the problem reduces to a simple quadratic programming and therefore can be efficiently solved;
- since $\|V\|_{1,1} = \sum_{j=1}^{K}\|\mathbf{v}_j\|_1$ we can rewrite

  $$R_{\lambda,\mathscr{C}}(V) = \frac{1}{2}\operatorname{tr}\left(V^\top \hat{\Sigma} V\right) - \operatorname{tr}\left(V^\top \hat{A} Z\right) + \lambda\|V\|_{1,1}$$
  $$= \sum_{j=1}^{K} \frac{1}{2}\mathbf{v}_j^\top \hat{\Sigma}\mathbf{v}_j - \mathbf{v}_j^\top \hat{A}\mathbf{z}_j + \lambda\|\mathbf{v}_j\|_1,$$

  therefore we need to solve K independent problems of size N, which reduces computational complexity and may also be implemented in parallel.

Ideally, we want to solve the following problem (note that the number of clusters K and the tuning parameter λ are fixed here)

$$F_\lambda(\mathscr{C}) \to \min_{\mathscr{C}}, \qquad F_\lambda(\mathscr{C}) = \min_V R_{\lambda,\mathscr{C}}(V).$$

We can employ a simple greedy procedure. In the beginning we initialize $\mathscr{C}^{(0)} = (l_1, \ldots, l_N)$ randomly, each label takes values $1, \ldots, K$. Then, at a step t we try to change one label of a node that reduces the risk the most. This means that we try all the clusterings in the nearest vicinity of a current solution $\mathscr{C}^{(t)}$, i.e.

$$\mathscr{C}^{(t+1)} = \arg\min_{d(\mathscr{C},\mathscr{C}^{(t)}) \le 1} F_\lambda(\mathscr{C}).$$

At each such step we would need to calculate $F_\lambda(\mathscr{C})$ for $\mathscr{O}(N(K-1))$ different candidates.

Remark 3.3. *In general, it is impossible to optimize arbitrary function $f(\mathscr{C})$ with respect to a clustering. For instance, there it is known that K-means is general NP-hard, however* *different solutions are widely used in practice, see Shindler et al. (2011) and Likas et al. (2003).*

To speed up the trials at of greedy procedure we utilize alternating minimization strategy. Suppose, at the beginning we initialize the clustering by $\mathscr{C}^{(0)}$ and compute the lasso solution $V^{(0)} = V_{\mathscr{C}^{(0)},t}$. When we want to update the clustering, we fix the matrix $V = V^{(t)}$ and solve the problem

$$R_{\mathscr{C},\lambda}(V) = \frac{1}{2}\operatorname{tr}(V^\top \hat{\Sigma} V) - \operatorname{tr}(V^\top \hat{A} Z_{\mathscr{C}}) + \lambda \|V\|_{1,1} \to \min_{\mathscr{C}},$$

where only the term $-\operatorname{tr}(V^\top \hat{A} Z_{\mathscr{C}})$ depends on $\mathscr{C}$. Minimizing by conducting a few steps of the greedy procedure we obtain the next clustering update $\mathscr{C}^{(t+1)}$. Then, we again update the V-factor by setting $V^{(t+1)} = V_{\mathscr{C}^{(t+1)},\lambda}$. We continue so until the clustering does not change or the number of iterations exceeds a certain limit. The pseudo code in Algorithm 1 summarizes this procedure.

Result: a pair $(\hat{\mathscr{C}}, \hat{V})$
initialize $\mathscr{C}^{(0)} = (l_1^{(0)}, \ldots, l_N^{(0)})$ randomly;
$t \leftarrow 0$;
while $t < max\_iter$ **do**
    update $\hat{V}^{(t)} \leftarrow \arg\min R_{\mathscr{C}^{(t)},\lambda}(V)$;
    for $i = 1, \ldots, N$ **do**
        for $l = 1, \ldots, N$ **do**
            consider candidate $\mathscr{C}' = (l_1^{(t)}, \ldots, l_{i-1}^{(t)}, l, l_{i+1}^{(t)}, \ldots, l_N^{(t)})$;
            $r_{il} \leftarrow -\operatorname{tr}(V^{(t)} \hat{A} Z_{\mathscr{C}'})$;
        end
    end
    $(i^*, l^*) = \arg\min r_{il}$;
    update $\mathscr{C}^{(t+1)} \leftarrow (l_1^{(t)}, \ldots, l_{i^*-1}^{(t)}, l^*, l_{i^*+1}^{(t)}, \ldots, l_N^{(t)})$;
    if $\mathscr{C}^{(t+1)} = \mathscr{C}^{(t)}$ **then**
        return $(\mathscr{C}^{(t)}, V^{(t)})$;
    else
        $t \leftarrow t + 1$;
    end
end

Algorithm 1: Alternating greedy clustering procedure.

3.2.4 Local consistency result

In this section we show the existence of a locally optimal solution in the neighbourhood of the true parameter with high probability. We call a clustering solution $\hat{\mathscr{C}}$ *locally optimal*, if the functional $F_\lambda(\cdot)$ has the minimum value at the point $\hat{\mathscr{C}}$ among it's nearest neighbours $d(\mathscr{C}, \hat{\mathscr{C}}) \leq 1$. In particular, Algorithm 1 obviously stops at such a solution. We first introduce some notation.

Notation

For a real vector $\mathbf{x} \in \mathbb{R}^d$ and $q \geq 1$ or $q = \infty$ denote ℓ_q-norm $\|\mathbf{x}\|_q = (|x_1|^q + \dots + |x_d|^q)^{1/q}$; for $q = 2$ we ignore the index, i.e. $\|\mathbf{x}\| = \|\mathbf{x}\|_2$; we also denote a pseudo-norm $\|\mathbf{x}\|_0 = \sum_i \mathbf{I}(x_i \neq 0)$. For a real matrix A denote $\|A\|_{\mathsf{F}} = \mathrm{tr}^{1/2}(A^\top A)$ is Frobenius norm. For $A \in \mathbb{R}^{d_1 \times d_2}$ denote $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_{\min(d_1,d_2)}(A)$ as it's non-trivial singular values. We will also refer to $\sigma_{\min}(A)$ as the least nontrivial eigenvalue, i.e. $\sigma_{\min}(A) = \sigma_{\min(d_1,d_2)}(A)$. Furthermore, we write $|||A|||_{\mathsf{op}} = \max_j \sigma_j(A)$ for spectral norm and $|||A|||_{\mathsf{F}} = \mathrm{tr}^{1/2}(A^\top A) = \left(\sum_{j=1}^{\min(p,q)} \sigma_j(A)^2\right)^{1/2}$ for Frobenius norm. Additionally, we introduce element-wise norms $\|A\|_{p,q}$ for $p, q \geq 1$ (including ∞) denotes ℓ_q norm of a vector composed of ℓ_p norms of rows of A, i.e. $\|A\|_{p,q} = \left(\sum_i \left(\sum_j |A_{ij}|^p\right)^{q/p}\right)^{1/q}$. Notice that $\|A\|_{2,2} = |||A|||_{\mathsf{F}}$.

Conditions

Here we describe the conditions that we need for the consistency result. The first condition concludes the requirements of Theorems 3.1 and 3.2.

Assumption 3.1. *There is some $\Theta^* \in \mathbb{R}^{N \times N}$ such that $|||\Theta^*|||_{\mathsf{op}} \leq \gamma$ for some $\gamma < 1$ and the time series Y_t follows* (3.3). *The innovations W_t are independent with $\mathsf{E} W_t = 0$ and* $\mathrm{Var}(W_t) = S$. *Moreover, each W_t is L-subgaussian.*

Furthermore, we impose structural assumptions onto the true parameter Θ^* described in Section 3.2.1.

Assumption 3.2. *The true VAR operator admits decomposition with K-clustering $\mathscr{C}^*$*

$$\Theta^* = Z_{\mathscr{C}^*} V^*,$$

and meets the following conditions:

1. $|||\Theta^*|||_{\mathsf{op}} = |||V^*|||_{\mathsf{op}} \le \gamma < 1$;

2. *cluster separation*

$$\sigma_{\min}([V^*]^\top \Sigma V^*) \ge a_0; \tag{3.5}$$

3. *sparsity: for each* $j = 1, \dots, K$ *the active set* $\Lambda_j = \operatorname{supp}(\mathbf{v}_j^*)$ *satisfies*

$$|\Lambda_j| \le s;$$

4. *significant active coefficients:*

$$|v_{ij}^*| \ge \tau_0 s^{-1/2}, \qquad i \in \Lambda_j, \qquad j = 1, \dots, K. \tag{3.6}$$

   Here each $\|\mathbf{v}_j^*\| \le 1$ *has (at most) s nonzero values, hence the normalization;*

5. *significant cluster sizes:*

$$\frac{\min_j |C_j^*|}{\max_j |C_j^*|} \ge \alpha, \qquad 0 < \alpha \le 1.$$

Notice that the condition (3.5) requires that the clusters appropriately separated, since it means in particular that each $\mathbf{v}_j^*$ is far enough from a linear combination of the rest. Another assumption is concerned with the population covariance Σ.

Assumption 3.3. *The covariance of* Y_t *reads as*

$$\Sigma = \sum_{k=0}^{\infty} (\Theta^*)^k S [(\Theta^*)^k]^\top,$$

where $S = \operatorname{Var}(W_t)$. *We impose the following assumptions onto this matrix.*

1. *bounded operator norm*

$$|||\Sigma|||_{\mathsf{op}} \le \sigma_{\max};$$

2. *restricted least eigenvalue*

$$\sigma_{\min}(\Sigma_{\Lambda_j,\Lambda_j}) \ge \sigma_{\min}, \qquad j = 1, \dots, K.$$

3. bounded $(1,1)$*-norm*

$$\|\Sigma^{-1}_{\Lambda_j,\Lambda_j}\|_{1,1} \le M, \qquad j = 1,\dots,K. \tag{3.7}$$

Remark 3.4. *Note, that we do not assume that the smallest eigenvalue of* Σ *is bounded away from zero, but only those corresponding to the small subsets of indices are. For sake of simplicity we additionally assume that the ratio*

$$\frac{\sigma_{\max}}{\sigma_{\min}} \le \kappa,$$

is bounded by some constant $\kappa \ge 1$.

Note also, that the bias term of the lasso term usually reads as $\hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}\mathbf{g}$ *with some* $\|\mathbf{g}\|_\infty \le 1$*, see Lemma A.1. We need* (3.7) *to control the sup-norm of this bias.*

Finally, we present the assumption that allows to control exact recovery of sparsity patterns for the lasso estimator.

Assumption 3.4. *For each* $j = 1,\dots,K$ *it holds*

$$\|\Sigma_{\Lambda_j^c,\Lambda_j}\Sigma^{-1}_{\Lambda_j,\Lambda_j}\|_{1,\infty} \le \frac{1}{4},$$

Remark 3.5. *The inequality* $\|\Sigma_{\Lambda_j^c,\Lambda_j}\Sigma^{-1}_{\Lambda_j,\Lambda_j}\|_{1,\infty} < 1$ *allows to derive exact recovery of the sparsity pattern at the LASSO procedure-step described above. In Section A.1 we show a straightforward extension of results from Tropp (2006) to the case with the presence of missing observations.*

Theorem 3.3. *Suppose, Assumptions 3.1-3.4 hold. There are constants* $c, C > 0$ *that depend on* L, γ *such that the following holds. Suppose,*

$$\sqrt{\frac{sn^*\log N}{Tp^2_{\min}}} \bigvee \sqrt{\frac{s\log N\log^2 T}{Tp^2_{\min}}} \le c, \tag{3.8}$$

where $n^* = \max_{j\le K}|C^*_j|$ *and, additionally,* $N \ge (C\alpha^2 \vee \kappa)K$*. Then, with probability at least* $1 - 1/N$ *for any* λ *satisfying*

$$C\sigma_{\max}\sqrt{\frac{\log N}{Tp^2_{\min}}} \le \lambda \le c\left(\kappa^{-4}(a_0^2/\sigma_{\max})K^{-2}s^{-1} \bigwedge \sigma_{\min}\tau_0 s^{-1}\right),$$

and, additionally, $\lambda \geq C\alpha^2 K/N$, there is a locally optimal solution $\hat{\mathscr{C}}$ satisfying

$$|||Z_{\hat{\mathscr{C}}}\hat{V}_{\hat{\mathscr{C}},\lambda}^{\top} - \Theta^*|||_{\mathsf{F}} \leq \left(3\sigma_{\min}^{-1}\sqrt{Ks} + \frac{C\gamma}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 K\sqrt{s}\right)\lambda\,.$$

Remark 3.6. *It also follows from the proof that under the assumptions of the theorem, the sparsity pattern of each vector is recovered precisely, i.e. we correctly identified the influencers for each cluster.*

Let us take a closer look at the condition (3.8). Under the cluster size restriction from Assumption 3.2 we have that all clusters have the size of order N/K, since

$$\alpha\frac{N}{K} \leq |C_j^*| \leq \alpha^{-1}\frac{N}{K}, \qquad j = 1,\ldots,K.$$

This means that, say if we ignore the missing observations, we only need

$$\frac{(sN/K)\log N}{T} \leq c(\alpha)$$

to hold, to be able to estimate the parameter. This means that once K is large enough the estimator works with the corresponding error. Notice that the ℓ_1-regularisation alone requires the number of the observations must be at least the number of edges times $\log N$, see Fan et al. (2009). In our setting the number of connections is up to Ns, so the condition reads as

$$\sqrt{\frac{sN\log N}{T}} \leq 1,$$

therefore our SoNIC model is an improvement in this regards.

According to the model, say if $N/K \geq \sqrt{T}$, the best available choice of tuning parameter is

$$\lambda^* = C\sigma_{\max}\sqrt{\frac{\log N}{Tp_{\min}^2}},$$

in which case the error of the estimator reads as

$$|||\hat{\Theta}_{\lambda^*} - \Theta^*|||_{\mathsf{F}} \lesssim K\sqrt{\frac{s\log N}{Tp_{\min}^2}},$$

which suggests some kind of tradeoff between small and large K.

3.3 Simulation study

Take $N = T = 100$ and $s = 1$, while K will be changing in a range 2..30. We are particularly interesting in capturing this effect that larger amount of clusters allows better estimation. For each $K = 2, \dots, 30$ we contruct the following matrix Θ^*,

- pick clusters C_j^* having approximately the same size $\frac{N}{K} \pm 1$;
- for each $j = 1, \dots, K$ set

$$\mathbf{v}_j^* = 0.5\mathbf{e}_j = (0, \dots, 0.5, \dots, 0)^\top,$$

  with a single nonzero value at the place j, so that $s = 1$.
- by construction we have,

$$|||\Theta^*|||_{\mathsf{op}} = |||V^*|||_{\mathsf{op}} = 0.5, \qquad |||\Theta^*|||_{\mathsf{F}} = |||V^*|||_{\mathsf{F}} = 0.5\sqrt{K}.$$

Furthermore we generate i.i.d. $W_{-19}, W_{-18}, \dots, W_T \sim \mathrm{N}(0, I)$ and set

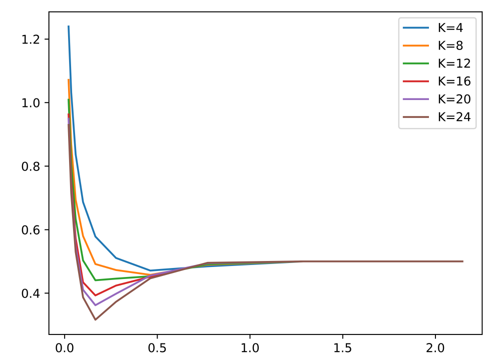
$$Y_t = \sum_{k=0}^{20} (\Theta^*)^k W_{t-k}, \qquad t = 1, \dots T,$$

where due to $0.5^{-20} \approx 10^{-6}$ the terms for $k > 20$ can easily be neglected. On Figure 3.3a we show the relative error $\mathsf{E}|||\hat{\Theta} - \Theta^*|||_{\mathsf{F}} / |||\Theta^*|||_{\mathsf{F}}$ along regularization paths for different choices of K. Picking the best λ we show the relative error against the number of clusters on Figure 3.3b. We also show the clustering error $\mathsf{E}d(\hat{\mathscr{C}}, \mathscr{C}^*)$ on Figure 3.3c depending on K. All expectations are estimated based on 20 simulations.

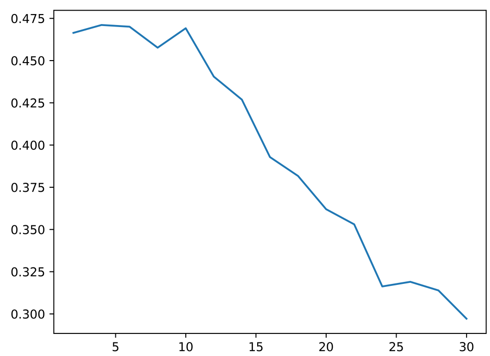
We conclude that the simulations confirm the following theoretical property of our estimator: the smaller the size of largest cluster, the better, while the total size of the network can be even as large as the number of observations.

3.4 Application to StockTwits sentiment

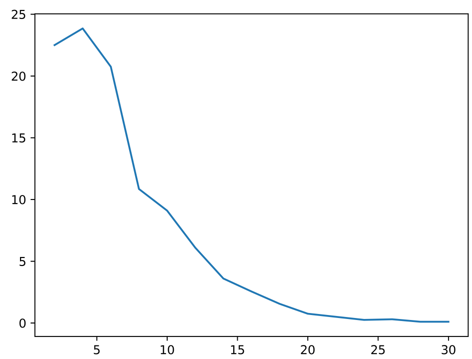
Here we present the results of experiment with two datasets described in Section 3.1. The first one contains daily average sentiment weights constructed from the messages containing



(a) Expected relative loss $\mathsf{E}\frac{\|\hat{\Theta}-\Theta^*\|_F}{\|\Theta^*\|_F}$ for different λ and $K = 4, 8, 12, 16, 20, 24$.



(b) Expected relative loss $\mathsf{E}\frac{\|\hat{\Theta}-\Theta^*\|_F}{\|\Theta^*\|_F}$ for the best λ and $K = 2, \ldots, 30$.



(c) Expected clustering error $\mathsf{E}d(\hat{\mathscr{C}}, \mathscr{C}^*)$ for the best λ and $K = 2, \ldots, 30$.

Figure 3.3 Simulation results for $N = T = 100$ and $s = 1$.

SoNIC_simulation_study

the cashtag '$AAPL' (Apple) and the second one from those containing the cashtag '$BTC.X' (Bitcoin.)

The missing observation model presented in Section 3.2.2 relies on persistent observation frequency with the same probability p_i over a time period under consideration. Moreover, since in Theorems 3.1 and 3.2 the amount of observations scales with the factor $p_{\min}^2$, we need to avoid the users whose p_i is too little. Based on these remarks we suggest the following preprocessing steps:

1. pick users with estimated probability $\hat{p}_i \geq 0.5$;
2. for each user left after step 1, pick the longest historical interval over which the user exhibits persistent probability of observation. One can look at a moving average estimation and ensure that for each window it remains within appropriate confidence interval;
3. take only users for whom the historical interval from step 2 is at least 50 days.

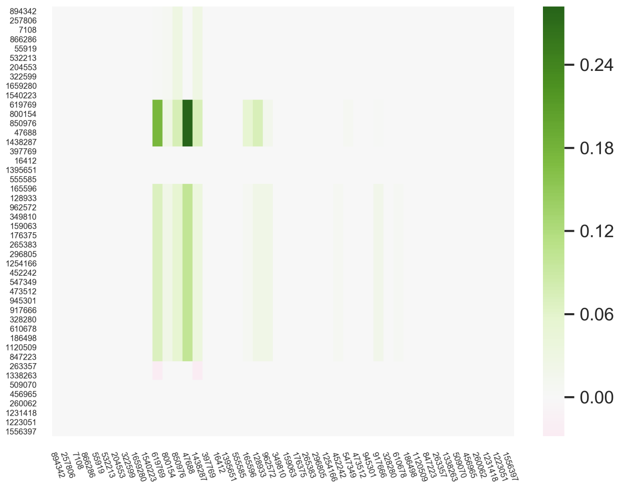
For AAPL dataset we are left with 46 users and 72 days, while for BTC we have 68 users and 52 days. The two datasets are visualized using heatmap in Figure 3.1.

We apply our SoNIC model to AAPL dataset with $\lambda = 0.05$ and $K = 6$. A heatmap visualisation for estimated matrix $\hat{\Theta}$ is presented in Figure 3.4a. From here we can identify that the most important users have identification number 47688, 619769, 850976 and 1438287[5]. For the BTC dataset we use $\lambda = 0.05$ and $K = 5$, the results presented in Figure 3.4b. The influencers are 1171931 and 1254166.

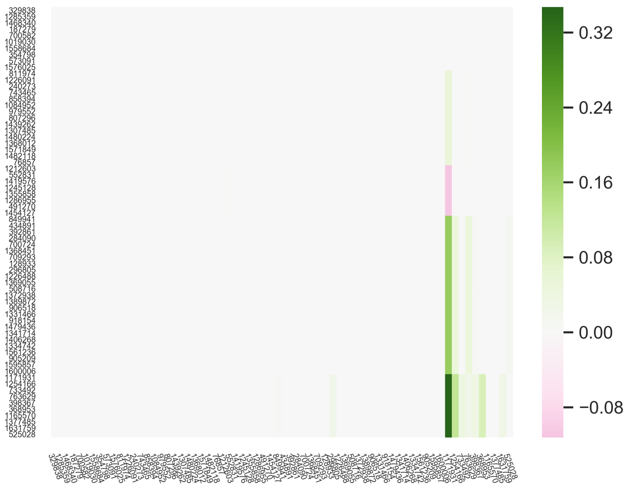
Remark 3.7. *Choosing the tuning parameter λ and the number of clusters K remains beyond the scope of this work. For this experiment we picked both numbers graphically: for λ based on the number of active columns with relatively small values, while for K we picked the smallest one for which there is no clusters that are much smaller than the others, as well as no clusters that are split into two or more. Development of a statistically-backed selection is left for further research.*

Let us point out some observations based on the results of this experiment. The first one is that for the Apple dataset we end up with users who have lots of followers, while from the Bitcoin dataset we have found two accounts that have moderate amount of followers and

[5]To access the page type https://stocktwits.com/user_id in the address line of a web browser.



(a) AAPL dataset with $N = 46$, $T = 72$ and $\lambda = 0.05$, $K = 5$.



(b) BTC dataset with $N = 68$, $T = 52$ and $\lambda = 0.05$, $K = 5$.

Figure 3.4 Estimated $\hat{\Theta}$ for AAPL and BTC datasets. The axes correspond to user id's and are rearranged with respect to the estimated clusterings.

SoNIC_AAPL_BTC

as it seems belong to companies that provide analytical tools for traders. We assume that it highlights the difference between two assets of different nature — a classical one and a cryptocurrency. Secondly, in both cases the "heaviest" users fall into the same cluster, though we do not provide any interpretation for this fact.

3.5 Proof of main result

This section is devoted to the proof of Theorem 3.3. We start with some preliminary lemmas and then proceed with the proof that consists of several steps. Following the ideas in Gribonval et al. (2015), the proof is based on explicit representation of the loss function.

We exploit the following simplified notation. Denote, $\mathbf{z}_j^* = \mathbf{z}_{C_j^*}$ to be the columns of $Z^* = Z_{\mathscr{C}^*}$ and we also denote $n_j^* = |C_j^*|$ for each $j = 1, \dots, K$. When the clustering $\mathscr{C} = (C_1, \dots, C_K)$ is clear from the context we will also write Z for $Z_{\mathscr{C}}$, $\mathbf{z}_j$ for $\mathbf{z}_{C_j}$, and $n_j = |C_j|$ for each $j = 1, \dots, K$. A vector $\mathbf{e}_j \in \mathbb{R}^d$ denotes a jth standard basis vector, i.e. jth element equal to one and the rest are zeros.

3.5.1 Preliminary lemmas

Lemma 3.1. *Suppose that C_j is such that $\|\mathbf{z}_{C_j} - \mathbf{z}_j^*\| \le 0.3$. Then,*

$$\frac{1}{1.1}|C_j^*| \le |C_j| \le 1.1|C_j^*|.$$

Proof. Suppose, $n_j = |C_j| > n_j^* = |C_j^*|$, then

$$r^2 = \|\mathbf{z}_j - \mathbf{z}_j^*\|^2 = 2 - \frac{2}{\sqrt{n_j n_j^*}}|C_j \cap C_j^*| \ge 2 - 2\sqrt{\frac{n_j^*}{n_j}},$$

since $|C_j \cap C_j^*| \le n_j^*$. Thus, $\sqrt{n_j} - \sqrt{n_j^*} \le (r^2/2)\sqrt{n_j}$, which due to $r \le 0.3$ implies by rearranging and taking square $n_j \le 1.1 n_j^*$.

If $n_j < n_j^*$ we have,

$$r^2 \ge \|\mathbf{z}_j - \mathbf{z}_j^*\|^2 = 2 - \frac{2|C_j \cap C_j'|}{\sqrt{n_j n_j^*}} \ge 2 - 2\sqrt{\frac{n_j}{n_j^*}},$$

and the fact that $r \le 0.3$ implies $n_j^* \le 1.1 n_j$.

$\square$

Lemma 3.2. *Let* $\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\| \le 0.3$. *Then,*

$$\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|_1 \le 1.55\sqrt{N_1}\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|^2 .$$

Proof. Let $N_j = |C_j|$ and $a = |C_1 \cap C_2|$, $b = |C_1 \setminus C_2|$, $c = |C_2 \setminus C_1|$, so that $N_1 = a + b$, $N_2 = a + c$, and $|C_1 \triangle C_2| = b + c$. We have,

$$\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|^2 = \left(\frac{1}{\sqrt{N_1}} - \frac{1}{\sqrt{N_2}}\right)^2 a + \frac{b}{N_1} + \frac{c}{N_2} \ge \frac{b}{N_1} + \frac{c}{N_2} .$$

On the other hand,

$$\begin{aligned}\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|_1 &= \left|\frac{1}{\sqrt{N_1}} - \frac{1}{\sqrt{N_2}}\right| a + \frac{b}{\sqrt{N_1}} + \frac{c}{\sqrt{N_2}} \\ &\le \left|\frac{1}{\sqrt{N_1}} - \frac{1}{\sqrt{N_2}}\right| a + \sqrt{N_1 \vee N_2}\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|^2 .\end{aligned}$$

Since $|N_1 - N_2| \le b + c$ we obviously have,

$$\begin{aligned}\left|\frac{1}{\sqrt{N_1}} - \frac{1}{\sqrt{N_2}}\right| a &= \frac{|N_1 - N_2| a}{\sqrt{(a+b)(a+c)}(\sqrt{a+b} + \sqrt{a+c})} \\ &\le \frac{(b+c)a}{\sqrt{N_1 \vee N_2}\sqrt{a}(2\sqrt{a})} \\ &\le \sqrt{N_1 \wedge N_2}\|\mathbf{z}_{C_1} - \mathbf{z}_{C_2}\|^2/2,\end{aligned}$$

and it is left to apply Lemma 3.1. $\square$

Lemma 3.3. *Suppose,* $\frac{\min_j n_j^*}{\max_j n_j^*} \ge \alpha$ *for some* $\alpha \in (0, 1]$ *and let* $\|\mathbf{z}_j - \mathbf{z}_j^*\| \le r$. *Suppose,* $r \le 0.3$. *Then,*

$$\|[Z^*]^\top (\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \le 3.05\alpha^{-1/2} r^2.$$

Proof. 1) We first consider the case $|C_j| = n_j^*$. It holds then

$$[\mathbf{z}_j^*]^\top (\mathbf{z}_j^* - \mathbf{z}_j) = \frac{1}{n_j^*}(n_j^* - |C_j \cap C_j^*|) = \frac{1}{n_j^*}|C_j^* \setminus C_j|.$$

Moreover, for each $k \neq j$ it holds

$$|[\mathbf{z}_k^*]^\top(\mathbf{z}_j^* - \mathbf{z}_j)| = |[\mathbf{z}_k^*]^\top \mathbf{z}_j| = \frac{1}{\sqrt{n_k^* n_j^*}}|C_k^* \cap C_j| \leq \frac{\alpha^{-1/2}}{n_j^*}|C_k^* \cap C_j|.$$

Summing up, we get

$$\begin{aligned}
\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 &\leq \frac{\alpha^{-1/2}}{n_j^*}\left(|C_j^* \setminus C_j| + \sum_{k \neq j}|C_k^* \cap C_j|\right) \\
&\leq \frac{\alpha^{-1/2}}{n_j^*}\left(|C_j^* \setminus C_j| + |C_j \setminus C_j^*|\right) \\
&= \frac{\alpha^{-1/2}}{n_j^*}|C_j \triangle C_j^*|.
\end{aligned}$$

It is left to notice that in the case $|C_j| = |C_j^*| = n_j^*$ we have exactly $\|\mathbf{z}_j - \mathbf{z}_j^*\|^2 = \frac{1}{n_j^*}|C_j \triangle C_j^*|$.

2) Suppose, $n_j = |C_j| > n_j^*$. Obviously, we can decompose $C_j = C_j' \cup B$ such that $|C_j'| = n_j^*$ and $B \cap C_j^* = \emptyset$. Setting $\mathbf{z}_j' = \mathbf{z}_{C_j'}$ we get by the above derivations that $\|[Z^*]^\top(\mathbf{z}_j' - \mathbf{z}_j^*)\|_1 \leq \alpha^{-1/2}\|\mathbf{z}_j' - \mathbf{z}_j^*\|^2$. Since $C_j' \cap C_j^* = C_j \cap C_j^*$ we can compare the distances

$$\|\mathbf{z}_j - \mathbf{z}_j^*\|^2 = 2 - \frac{2}{\sqrt{n_j n_j^*}}|C_j \cap C_j^*| > 2 - \frac{2}{n_j^*}|C_j \cap C_j^*| = \|\mathbf{z}_j' - \mathbf{z}_j^*\|^2.$$

Taking the remainder $\mathbf{b} = \mathbf{z}_j - \mathbf{z}_j'$ we have that

$$b_i = \begin{cases} {n_j}^{-1/2} - (n_j^*)^{-1/2}, & i \in C_j', \\ {n_j}^{-1/2}, & i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Setting $d = n_j - n_j^* = |B|$ it is easy to obtain $|{n_j}^{-1/2} - (n_j^*)^{-1/2}| \leq \frac{d}{n_j}\frac{1}{\sqrt{n_j^*}}$. Thus, we get

$$\begin{aligned}\sum_{k=1}^{K} |[\mathbf{z}_k^*]^\top \mathbf{b}| &\leq \sum_{i=1}^{k} \frac{1}{\sqrt{n_k^*}} \left(\frac{d}{n_j} \frac{1}{\sqrt{n_j^*}} |C_j' \cap C_k^*| + |B \cap C_k^*| \frac{1}{\sqrt{n_j}} \right) \\ &\leq \frac{\alpha^{-1/2} d}{n_j^* n_j} |C_j'| + \frac{\alpha^{-1/2}}{\sqrt{n_j^* n_j}} d \\ &< \frac{2\alpha^{-1/2} d}{\sqrt{n_j n_j^*}}.\end{aligned}$$

We show that the latter is at most $2.05\alpha^{-1/2} r^2$. Indeed, it is not hard to show that from $n_j \leq 1.1 n_j^*$ (see Lemma 3.1) it follows

$$\frac{n_j - n_j^*}{\sqrt{n_j n_j^*}} \leq 2.05 \left(1 - \frac{n_j^*}{\sqrt{n_j n_j^*}} \right) \leq 2.05 \times \frac{r^2}{2},$$

thus $\|[Z^*]^\top (\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \leq 3.05\alpha^{-1/2} r^2$ and the result follows.

3) The case $n_j < n_j^*$ can be resolved similarly to the previous one. Since $|C_j^* \setminus C_j| \geq n_j^* - n_j$ we can pick a subset $B \subset C_j^* \setminus C_j$ of size $d = n_j^* - n_j$ and set $C_j' = B \cup C_j$ with $|C_j'| = n_j^*$; set also $\mathbf{z}_j' = \mathbf{z}_{C_j'}$. Then, we have

$$\|\mathbf{z}_j' - \mathbf{z}_j^*\|^2 = 2 - 2\frac{|C_j' \cap C_j^*|}{n_j^*} \leq 2 - \frac{2|C_j \cap C_j'|}{\sqrt{n_j n_j^*}} = \|\mathbf{z}_j - \mathbf{z}_j^*\|^2,$$

and it is not hard to derive that $\|\mathbf{z}_j' - \mathbf{z}_j^*\|^2 \leq \|\mathbf{z}_j - \mathbf{z}_j^*\|^2$. Thus, by the first part of this proof it holds $\|[Z^*]^\top (\mathbf{z}_j' - \mathbf{z}_j^*)\|_1 \leq \alpha^{-1/2} r^2$. Setting $\mathbf{b} = \mathbf{z}_j' - \mathbf{z}_j$ we have,

$$b_i = \begin{cases} (n_j^*)^{-1/2} - {n_j}^{-1/2}, & i \in C_j, \\ n_j^{*-1/2}, & i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|n_j^{-1/2} - (n_j^*)^{-1/2}| \le \frac{d}{n_j^*}\frac{1}{\sqrt{n_j}}$ we obtain,

$$\begin{aligned}
\sum_{k=1}^{K} |[\mathbf{z}_k^*]^\top \mathbf{b}| &\le \sum_{i=1}^{k} \frac{1}{\sqrt{n_k^*}} \left(\frac{d}{n_j^*}\frac{1}{\sqrt{n_j}} |C_j \cap C_k^*| + |B \cap C_k^*| \frac{1}{\sqrt{n_j^*}} \right) \\
&\le \frac{\alpha^{-1/2} d}{(n_j^*)^{3/2} n_j^{1/2}} |C_j| + \frac{\alpha^{-1/2}}{n_j^*} d \\
&< \frac{2\alpha^{-1/2} d}{n_j^*}.
\end{aligned}$$

It is left to notice that

$$r^2 \ge 2 - \frac{2n_j}{\sqrt{n_j n_j^*}} = \frac{2(\sqrt{n_j^*} - \sqrt{n_j})}{\sqrt{n_j}} = \frac{2(n_j^* - n_j)}{n_j^* + \sqrt{n_j n_j^*}} \ge \frac{2d}{2n_j^*},$$

therefore $\|[Z^*]^\top \mathbf{b}\|_1 \le 2\alpha^{-1/2} r^2$, thus $\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \le 3\alpha^{-1/2} r^2$. □

Lemma 3.4. *Let $r = \|\|Z_{\mathscr{C}} - Z^*\|\|_{\mathsf{F}}$ and suppose that $r \le 0.3$. Then $\|\|P_{\mathscr{C}} - P_{\mathscr{C}^*}\|\|_{\mathsf{F}}^2 \ge 2r^2(1 - 10\alpha^{-1} r^2)$.*

Proof. Denote $\mathbf{z}_j = \mathbf{z}_{C_j}$ and $r_j = \|\mathbf{z}_j - \mathbf{z}_j^*\|$. It holds,

$$\|\|P_{\mathscr{C}} - P_{\mathscr{C}^*}\|\|_{\mathsf{F}}^2 = 2K - 2\operatorname{tr}(P_{\mathscr{C}} P_{\mathscr{C}^*}) = 2K - \sum_{j,k} (\mathbf{z}_j^\top \mathbf{z}_k^*)^2.$$

Notice, that $2\mathbf{z}_j^\top \mathbf{z}_j^* = 2 - \|\mathbf{z}_j\|^2 - \|\mathbf{z}_j^*\|^2 + 2\mathbf{z}_j^\top \mathbf{z}_j^* = 2 - \|\mathbf{z}_j - \mathbf{z}_j^*\|^2$, i.e. $\mathbf{z}_j^\top \mathbf{z}_j^* = 1 - r_j^2/2$. In particular, $1 - (\mathbf{z}_j^\top \mathbf{z}_j^*)^2 = r_j^2 - r_j^4/4$, whereas $([\mathbf{z}_j^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*))^2 = r_j^4/4$. Since we additionally have $[\mathbf{z}_k^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*) = [\mathbf{z}_k^*]^\top \mathbf{z}_j$ for $k \ne j$, it holds

$$\begin{aligned}
2K - 2\sum_{j,k} (\mathbf{z}_j^\top \mathbf{z}_k^*)^2 &= 2\sum_j r_j^2 - r_j^4/4 - 2\sum_j \sum_{k \ne j} \left([\mathbf{z}_k^*]^\top (\mathbf{z}_j - \mathbf{z}_j^*)\right)^2 \\
&= 2r^2 - 2\sum_{j,k} \left([\mathbf{z}_k^*]^\top (\mathbf{z}_j - \mathbf{z}_j^*)\right)^2 \\
&= 2r^2 - 2\sum_j \|[Z^*]^\top (\mathbf{z}_j - \mathbf{z}_j^*)\|^2
\end{aligned}$$

By Lemma 3.3 we have for each $j = 1, \ldots, K$

$$\|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\| \le \|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \le 3.05\alpha^{-1/2} r_j^2,$$

therefore

$$\sum_j \|[Z^*]^\top(\mathbf{z}_j - \mathbf{z}_j^*)\|^2 \le 10\alpha^{-1}\sum_j r_j^4 \le 10\alpha^{-1} r^4,$$

thus inequality follows. □

Lemma 3.5. *Let* C, C' *be such that* $|C \triangle C'| = 1$. *Then* $\|\mathbf{z}_C - \mathbf{z}_{C'}\|^2 \le \frac{2}{|C| \vee |C'|}$.

Proof. Suppose, $|C'| > |C|$ then $C' = C \cup \{a\}$ and denoting $n = |C|$ we have

$$\|\mathbf{z}_C - \mathbf{z}_{C'}\|^2 = n\left(\sqrt{\frac{1}{n+1}} - \sqrt{\frac{1}{n}}\right)^2 + \frac{1}{n+1} = \frac{(\sqrt{n+1} - \sqrt{n})^2 + 1}{n+1} \le \frac{2}{n+1}.$$

□

3.5.2 Proof of Theorem 3.3

The proof consists of several steps, each represented by a separate lemma.

Lemma 3.6. *Suppose, Assumption 3.1 holds and let* $N \ge 2$. *There is a constant* $C = C(\gamma, L)$, *so that if*

$$\frac{s \log N \log^2 T}{T p_{\min}^2} \le \frac{1}{3},$$

then with probability at least $1 - 1/N$ *and for with* $\Delta_1 = C\sigma_{\max}\sqrt{\frac{\log N}{T p_{\min}^2}}$ *the following inequalities take place for each* $j = 1, \dots, K$

- $$\|\hat{A} - A\|_{\infty,\infty} \le \Delta_1, \qquad \|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\|_{\infty,\infty} \le \sigma_{\min}^{-1}\Delta_1; \tag{3.9}$$

- $$\|(\hat{A} - A)\mathbf{z}_j^*\|_\infty \le \Delta_1, \qquad \|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j^*\|_\infty \le \sigma_{\min}^{-1}\Delta_1; \tag{3.10}$$

- $$\|\hat{\Sigma} - \Sigma\|_{\infty,\infty} \le \Delta_1, \qquad \|(\hat{\Sigma}_{\Lambda_j,\cdot} - \Sigma_{\Lambda_j,\cdot})\mathbf{v}_j^*\|_\infty \le \Delta_1; \tag{3.11}$$

- $$\|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{\Sigma}_{\Lambda_j,\cdot} - \Sigma_{\Lambda_j,\cdot})\mathbf{v}_j^*\|_\infty \le \sigma_{\min}^{-1}\Delta_1; \tag{3.12}$$

- $$\|\hat{\Sigma}_{\Lambda_j,\Lambda_j} - \Sigma_{\Lambda_j,\Lambda_j}\|_{\mathsf{op}} \le \sqrt{s}\Delta_1. \tag{3.13}$$

Proof. By Theorem 3.2 it holds for any pair $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ with $\|\mathbf{a}\| \leq 1$, $\|\mathbf{b}\| \leq 1$ it holds probability $\geq 1 - N^{-m}$,

$$|\mathbf{a}^\top(\hat{A} - A)\mathbf{b}| \leq C\sigma_{\max} \left(\sqrt{\frac{(m+1)\log N}{T p_{\min}^2}} \bigvee \frac{(m+1)\log N \log T}{T p_{\min}^2} \right).$$

Suppose for a moment that m is such that

$$\sqrt{\frac{(m+1)s\log N}{T p_{\min}^2}} \log T \leq 1, \tag{3.14}$$

so that we can neglect the second term. Set,

$$A_0 = \{(\mathbf{e}_i, \mathbf{e}_{i'}) : \, i, i' \leq N\}, \qquad B_0 = \{(\mathbf{e}_i, \mathbf{z}_l^*) : \, i \leq N, l \leq K\},$$

as well as for each $j = 1, \dots, K$

$$A_j = \{(\sigma_{\min}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{e}_i, \mathbf{e}_{i'}) : \, i \in \Lambda_j, i' \leq N\},$$
$$B_j = \{(\sigma_{\min}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{e}_i, \mathbf{z}_l^*) : \, i \in \Lambda_j, l \leq K\}.$$

Obviously we have $|A_0| \leq N^2, |B_0| \leq NK$ and $|A_j| \leq sN, |B_j| \leq sK$ for $j = 1, \dots, N$, so since $s, K \leq N$ together they have not more than $4N^3$ pairs of vectors $(\mathbf{a}, \mathbf{b})$, each having norm bounded by one. Taking a union bound, we have that the inequalities (3.9) and (3.10) hold with probability at least $1 - 4N^{3-m}$. By analogy, we can show that (3.11) and (3.12) hold with probability at least $1 - 4N^{3-m}$.

As for the last inequality, for each $j = 1, \dots, K$ pick $P_j = \sum_{i \in \Lambda_j} \mathbf{e}_i \mathbf{e}_i^\top$, i.e. projectors onto the subspace of vectors supported on Λ_j. Then by Theorem 3.1 it holds with probability at least $1 - KN^{-m}$ for each $j = 1, \dots, K$ (taking into account (3.14))

$$|||\hat{\Sigma}_{\Lambda_j,\Lambda_j} - \Sigma_{\Lambda_j,\Lambda_j}|||_{\mathsf{op}} = |||P_j(\hat{\Sigma} - \Sigma)P_j|||_{\mathsf{op}} \leq C\sigma_{\max}\sqrt{\frac{s(m+1)\log N}{T p_{\min}^2}}.$$

The total probability will be at least $1 - 8N^{3-m} - KN^{-m}$, which is at least $1 - 1/N$ whenever $m \geq 7$ and $N \geq 2$.

□

In the following we apply technique from Gribonval et al. (2015). Suppose, that the lasso solution $\hat{\mathbf{v}}_j$ for a given clustering $\mathscr{C}$ is not only supported exactly on Λ_j, but the signs are matching those of the true $\mathbf{v}_j^*$. Then, $\|\hat{\mathbf{v}}_j\|_1 = \bar{\mathbf{s}}_j^\top (\hat{\mathbf{v}}_j)_{\Lambda_j}$. Therefore, we can write

$$
\begin{aligned}
(\hat{\mathbf{v}}_j)_{\Lambda_j} &= \arg\min_{\mathbf{v}\in\mathbb{R}^{\Lambda_j}} \frac{1}{2}\mathbf{v}^\top \hat{\Sigma}_{\Lambda_j,\Lambda_j}\mathbf{v} - \mathbf{v}^\top \hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j + \lambda \bar{\mathbf{s}}_j^\top \mathbf{v} \\
&= \hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1}(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\bar{\mathbf{s}}_j),
\end{aligned}
$$

and plugging this solution into the risk function we get that $F_\lambda(\mathscr{C}) = \Phi_\lambda(\mathscr{C})$, where the latter is defined explicitly

$$
\Phi_\lambda(\mathscr{C}) = -\frac{1}{2}\sum_{j=1}^{K}(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\bar{\mathbf{s}}_j)^\top \hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1}(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\bar{\mathbf{s}}_j).
$$

Lemma 3.7. *Suppose, the inequalities* (3.9)–(3.13) *take place. Assume,*

$$
s\Delta_1 \le 1/16, \qquad 12\Delta_1 \le \lambda \le \frac{\sigma_{\min}}{4}\tau_0 s^{-1}. \tag{3.15}
$$

Then, for each $\mathscr{C} = (C_1,\ldots,C_K)$ satisfying

$$
\max_j \|\mathbf{z}_{C_j} - \mathbf{z}_{C_j^*}\| \le 0.3 \wedge 0.22\sqrt{\left(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1\right)^{-1}\lambda} \tag{3.16}
$$

it holds

$$
\|\hat{V}_{\lambda,\mathscr{C}} - V^*\|_{\mathsf{F}} \le 3\sigma_{\min}^{-1}\sqrt{Ks}\lambda,
$$

and the equality $F_\lambda(\mathscr{C}) = \Phi_\lambda(\mathscr{C})$ takes place.

Proof. Taking into account $Z^\top Z = I_K$, it holds

$$
\begin{aligned}
R_{\lambda,\mathscr{C}}(V) =& \frac{1}{2}\operatorname{tr}\left(V^\top \hat{\Sigma} V\right) - \operatorname{tr}\left(V^\top \hat{A} Z\right) + \lambda\|V\|_{1,1} \\
=& \sum_{j=1}^{K} \frac{1}{2}\mathbf{v}_j^\top \hat{\Sigma}\mathbf{v}_j - \mathbf{v}_j^\top \hat{A}\mathbf{z}_j + \lambda\|\mathbf{v}_j\|_1,
\end{aligned}
$$

so that the optimization problem separates into K independent subproblems. Solving each of the problems

$$
\frac{1}{2}\mathbf{v}_j^\top \hat{\Sigma}\mathbf{v}_j - \mathbf{v}_j^\top \hat{A}\mathbf{z}_j + \lambda\|\mathbf{v}_j\|_1 \to \min_{\mathbf{v}_j}
$$

corresponds to Corollary A.1 with $\hat{D} = \hat{\Sigma}$ and $\hat{\mathbf{c}} = \hat{A}\mathbf{z}_j$, whereas the "true" version of the problem corresponds to $\bar{D} = \Sigma$ and $\bar{\mathbf{c}} = A\mathbf{z}_j^* = \Sigma(\Theta^*)^\top \mathbf{z}_j^* = \Sigma \mathbf{v}_j^*$. We need to control the differences between $\hat{\mathbf{c}}$ and $\bar{\mathbf{c}}$, and between $\hat{D}$ and $\bar{D}$. It holds,

$$\|\hat{A}\mathbf{z}_j - A\mathbf{z}_j^*\|_\infty \leq \|A(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty + \|(\hat{A} - A)\mathbf{z}_j^*\|_\infty + \|(\hat{A} - A)(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty .$$

Since $A = \Sigma V^* [Z^*]^\top$, we bound the first term using Lemma 3.3

$$\|A(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty \leq \|\Sigma V^*\|_{\infty,\infty} \|[Z^*]^\top (\mathbf{z}_j - \mathbf{z}_j^*)\|_1 \leq 3.05\alpha^{-1/2} \|\Sigma V^*\|_{\infty,\infty} r_j^2.$$

The second term is bounded by Δ_1, whereas the fourth term satisfies

$$\|(\hat{A} - A)(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty \leq \|\hat{A} - A\|_{\infty,\infty} \|\mathbf{z}_j - \mathbf{z}_j^*\|_1 \leq 1.55\Delta_1 \sqrt{n^*} r_j^2,$$

where we also used Lemma 3.2. Summing up we get,

$$\|\hat{\mathbf{c}} - \mathbf{c}\|_\infty \leq 1.55(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n_j^*}\Delta_1) r_j^2 + \Delta_1 .$$

Similarly, we bound $\|\Sigma_{\Lambda_j,\Lambda_j}(\hat{\mathbf{c}}_{\Lambda_j} - \bar{\mathbf{c}}_{\Lambda_j})\|_\infty$ as follows

$$\begin{aligned}
\|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - A_{\Lambda_j,\cdot}\mathbf{z}_j^*)\|_\infty \leq & \|\Sigma^{-1}_{\Lambda_j,\Lambda_j} A(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty + \|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j^*\|_\infty \\
& + \|\Sigma^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty \\
\leq & \|\Sigma^{-1}_{\Lambda_j,\Lambda_j} A(\mathbf{z}_j - \mathbf{z}_j^*)\|_\infty + 1.55\sigma_{\min}^{-1}\Delta_1\sqrt{n^*} r_j^2 + \sigma_{\min}^{-1}\Delta_1 \\
\leq & 1.55\sigma_{\min}^{-1}(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n_j^*}\Delta_1) r_j^2 + \sigma_{\min}^{-1}\Delta_1
\end{aligned}$$

To sum up, Corollary A.1 is applied with

$$\begin{aligned}
\delta_c =& 1.55(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1) r_j^2 + \Delta_1, \\
\delta_c' =& 1.55\sigma_{\min}^{-1}(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1) r_j^2 + \sigma_{\min}^{-1}\Delta_1 \\
\delta_D =& \Delta_1, \qquad \delta_D' = \Delta_1, \qquad \delta_D'' = \sigma_{\min}^{-1}\Delta_1.
\end{aligned}$$

It requires the conditions,

$$3(1.55(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1) r_j^2 + 2\Delta_1) \leq \lambda, \qquad s\Delta_1 \leq \frac{1}{16},$$

and due to the fact that $\|D^{-1}_{\Lambda_j,\Lambda_j}\|_{1,\infty} \le \sqrt{s}|||D^{-1}_{\Lambda_j,\Lambda_j}|||_{\mathsf{op}}$ and Assumption 3.6,

$$2\sigma_{\min}^{-1}(1.55(2\sigma_{\max}\alpha^{-1/2}+\sqrt{n^*}\Delta_1)r_j^2+2\Delta_1+\sqrt{s}\lambda) < \tau_0 s^{-1/2},$$

which are not hard to derive from the given inequalities. All this that $\hat{\mathbf{v}}_j$ is supported on Λ_j and the solution satisfies

$$(\hat{\mathbf{v}}_j)_{\Lambda_j} = \hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}\left(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right),$$

and the corresponding minimum is equal to

$$\frac{1}{2}\hat{\mathbf{v}}_j^\top\hat{\Sigma}\hat{\mathbf{v}}_j^\top - \hat{\mathbf{v}}_j^\top\hat{A}\mathbf{z}_j + \lambda(\hat{\mathbf{v}}_j)_{\Lambda_j}^\top\mathbf{s}_j^* = -\frac{1}{2}\left(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right)^\top\hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}\left(\hat{A}_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right).$$

Summing up we get the corresponding expression for $F_\lambda(\mathscr{C})$. Moreover, we have

$$\begin{aligned}\|\hat{\mathbf{v}}_j - \mathbf{v}_j^*\| \le& 2\sqrt{s}\left\{2\Delta_1 + 1.55(2\sigma_{\max}\alpha^{-1}+\sqrt{n^*}\Delta_1)r_j^2+\lambda\right\}\\ \le& 2\sigma_{\min}^{-1}\sqrt{s}\left(\frac{\lambda}{6}+\frac{1.55\lambda}{20}+\lambda\right)\\ \le& 3\sigma_{\min}^{-1}\sqrt{s}\lambda,\end{aligned}$$

and together it provides a bound on $|||\hat{V}_{\lambda,\mathscr{C}} - V^*|||_{\mathsf{F}}$. □

Consider the function,

$$\bar{\Phi}_\lambda(\mathscr{C}) = -\frac{1}{2}\sum_{j=1}^{k}\left(A_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right)^\top\Sigma^{-1}_{\Lambda_j,\Lambda_j}\left(A_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right).$$

The growth of this function as $\mathscr{C}$ recedes from the true clustering $\mathscr{C}^*$ is controlled by the following lemma.

Lemma 3.8. *Suppose, $\mathscr{C}$ is some clustering such that $r = |||Z_{\mathscr{C}} - Z^*|||_{\mathsf{F}} \le 0.3$. Then,*

$$\bar{\Phi}_\lambda(\mathscr{C}) - \bar{\Phi}_\lambda(\mathscr{C}^*) \ge \frac{a_0}{2}r^2(1-10\alpha^{-1}r^2) - \lambda\sqrt{Ks}|||V^*|||_{\mathsf{F}}r.$$

Proof. Denoting $\bar{\Phi}_0(\mathscr{C}) = -\frac{1}{2}\sum_{j=1}^{k} \mathbf{z}_j^\top \hat{A}_{\Lambda_j,\cdot}^\top \hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1} \hat{A}_{\Lambda_j,\cdot} \mathbf{z}_j$ (which indeed corresponds to $\lambda = 0$), we have the decomposition

$$\bar{\Phi}_\lambda(\mathscr{C}) - \bar{\Phi}_\lambda(\mathscr{C}^*) = \bar{\Phi}_0(\mathscr{C}) - \bar{\Phi}_0(\mathscr{C}^*) - \lambda \sum_{j=1}^{K} [\mathbf{s}_j^*]^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}(\mathbf{z}_j - \mathbf{z}_j^*).$$

Let us first deal with the term $\bar{\Phi}_0(\mathscr{C}) - \bar{\Phi}_0(\mathscr{C}^*)$. Note that since $[\mathbf{v}_j^*]_{\Lambda_j} = \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}\mathbf{z}_j^*$, we have

$$\bar{\Phi}_0(\mathscr{C}^*) = -\frac{1}{2}\sum_{j=1}^{K} [\mathbf{v}_j^*]^\top \Sigma \mathbf{v}_j^* = -\frac{1}{2}\operatorname{tr}([V^*]^\top \Sigma V^*) = -\frac{1}{2}\operatorname{tr}(\Theta^* \Sigma [\Theta^*]^\top).$$

whereas

$$\bar{\Phi}_0(\mathscr{C}) = \min_{V=[\mathbf{v}_1,\ldots,\mathbf{v}_k]} \frac{1}{2}\operatorname{tr}(V^\top \Sigma V) - \operatorname{tr}(V^\top A Z)$$

where the minimum is taken s.t. the restrictions $\operatorname{supp}(\mathbf{v}_j) \subset \Lambda_j$. Dropping the restrictions we get,

$$\begin{aligned}
\bar{\Phi}_0(\mathscr{C}) - \bar{\Phi}_0(\mathscr{C}^*) &\geq \min_V \frac{1}{2}\operatorname{tr}(V^\top \Sigma V) - \operatorname{tr}(V^\top A Z) + \frac{1}{2}\operatorname{tr}(\Theta^* \Sigma [\Theta^*]^\top) \\
&= \min_V \frac{1}{2}|||Z V^\top \Sigma^{1/2}|||_{\mathsf{F}}^2 - \operatorname{tr}(Z V^\top \Sigma [\Theta^*]^\top) + |||\Theta^* \Sigma^{1/2}|||_{\mathsf{F}}^2 \\
&= \min_V \frac{1}{2}|||(Z V^\top - \Theta^*)\Sigma^{1/2}|||_{\mathsf{F}}^2.
\end{aligned}$$

It is not hard to calculate that the minimum is attained for $V = [\Theta^*]^\top Z$ and therefore

$$\bar{\Phi}_0(\mathscr{C}) - \bar{\Phi}_0(\mathscr{C}^*) \geq \frac{1}{2}|||(ZZ^\top - I)\Theta^* \Sigma^{1/2}|||_{\mathsf{F}}^2 \geq \frac{a_0}{2}|||(ZZ^\top - I)Z^*|||_{\mathsf{F}}^2,$$

where the latter follows using $\Theta^* = Z^*[V^*]^\top$ and from the fact that $\lambda_{\min}([V^*]^\top \Sigma V^*) \geq \sigma_0$. Moreover,

$$\begin{aligned}
|||(ZZ^\top - I)Z^*|||_{\mathsf{F}}^2 &= \operatorname{tr}((P_{\mathscr{C}} - I)P_{\mathscr{C}^*}(P_{\mathscr{C}} - I)) = \operatorname{tr}(P_{\mathscr{C}^*}) - \operatorname{tr}(P_{\mathscr{C}} P_{\mathscr{C}^*}) \\
&= \frac{1}{2}|||P_{\mathscr{C}} - P_{\mathscr{C}^*}|||_{\mathsf{F}}^2,
\end{aligned}$$

where we used the fact that $\operatorname{tr}(P_{\mathscr{C}}) = \operatorname{tr}(P_{\mathscr{C}^*}) = K$. It is left to recall the result of Lemma 3.4, so that we get

$$\bar{\Phi}_0(\mathscr{C}) - \bar{\Phi}_0(\mathscr{C}^*) \geq \frac{a_0 r^2}{2}(1 - 10\alpha^{-1} r^2).$$

As for the linear term, it holds

$$\left(\sum_{j=1}^{K}[\mathbf{s}_j^*]^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}(\mathbf{z}_j-\mathbf{z}_j^*)\right)^2 \le \left(\sum_{j=1}^{K}\|[\mathbf{s}_j^*]^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}\|^2\right) r^2$$

Since $A=\Sigma[\Theta^*]^\top$, we have $A_{\Lambda_j,\cdot}^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^* = \Theta^*\Sigma_{\cdot,\Lambda_j}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^*$. Denote, $\mathbf{x}=\Sigma_{\cdot,\Lambda_j}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^*$, then we have $\mathbf{x}_{\Lambda_j}=\mathbf{s}_j$ and $\|\mathbf{x}_{\Lambda_j}\|_\infty = 1$. Moreover, by the ERC property

$$\|\mathbf{x}_{\Lambda_j^c}\|_\infty = \|\Sigma_{\Lambda_j^c,\Lambda_j}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j\|_\infty \le \|\Sigma_{\Lambda_j^c,\Lambda_j}\Sigma_{\Lambda_j,\Lambda_j}^{-1}\|_{1,\infty} \le 1/2.$$

We have

$$\|A_{\Lambda_j,\cdot}^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^*\|^2 = \|\sum \mathbf{z}_j^*[\mathbf{v}_j^*]^\top \mathbf{x}\|^2 = \sum_{k=1}^{K}|[\mathbf{v}_k^*]^\top \mathbf{x}|^2,$$

where, since $\mathbf{v}_k^*$ is supported on Λ_k of size at most s,

$$|[\mathbf{v}_k^*]^\top \mathbf{x}| \le \|\mathbf{v}_k^*\|_1\|\mathbf{x}\|_\infty \le \sqrt{s}\|\mathbf{v}_k^*\|.$$

Summing up we get $\|A_{\Lambda_j,\cdot}^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1}\mathbf{s}_j^*\|^2 \le s\|\!|V^*|\!\|_{\mathsf{F}}^2$, so that

$$\left|\sum_{j=1}^{K}[\mathbf{s}_j^*]^\top \Sigma_{\Lambda_j,\Lambda_j}^{-1} A_{\Lambda_j,\cdot}(\mathbf{z}_j-\mathbf{z}_j^*)\right| \le \sqrt{Ks}\|\!|V^*|\!\|_{\mathsf{F}} r.$$

The lemma now follows from the two terms put together. □

The next step is to bound the difference $\Phi_\lambda(\mathscr{C})-\bar{\Phi}_\lambda(\mathscr{C})$ uniformly in the neighbourhood of $\mathscr{C}^*$.

Lemma 3.9. *Suppose that the inequalities* (3.9)–(3.13) *hold and let*

$$\Delta_1 \le \sigma_{\min}/(2\sqrt{s}) \vee \frac{\lambda}{12}, \qquad \sigma_{\max}/\sigma_{\min} \le n^*, \qquad \lambda \le \sigma_{\min}s^{-1}$$

Let some $r \le 0.3$ *satisfies* $\sqrt{sn^*}\Delta_1 r^2 \le \sigma_{\max}$. *Then,*

$$\begin{aligned}\sup_{\|\!|Z-Z^*|\!\|_{\mathsf{F}}\le r} &|\Phi_\lambda(\mathscr{C})-\bar{\Phi}_\lambda(\mathscr{C})-\Phi_\lambda(\mathscr{C}^*)+\bar{\Phi}_\lambda(\mathscr{C}^*)| \\ &\le 4\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}\|\!|V^*|\!\|_{\mathsf{F}} + \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1 r + 15\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 r^2.\end{aligned}$$

Proof. Denote,

$$\tilde{\Phi}_\lambda(\mathscr{C}) = -\frac{1}{2}\sum_{j=1}^{K}\left(A_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right)^\top \hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1}\left(A_{\Lambda_j,\cdot}\mathbf{z}_j - \lambda\mathbf{s}_j^*\right),$$

so that we have

$$\begin{aligned}
&|\tilde{\Phi}_\lambda(\mathscr{C}) - \bar{\Phi}_\lambda(\mathscr{C}) - \tilde{\Phi}_\lambda(\mathscr{C}^*) + \bar{\Phi}_\lambda(\mathscr{C}^*)| \\
&\qquad \le \frac{1}{2}\sum_{j=1}^{K}\left|\left(A_{\Lambda_j,\cdot}(\mathbf{z}_j+\mathbf{z}_j^*) - 2\lambda\mathbf{s}_j^*\right)^\top (\hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1} - \Sigma_{\Lambda_j,\Lambda_j}^{-1})A_{\Lambda_j,\cdot}(\mathbf{z}_j-\mathbf{z}_j^*)\right|
\end{aligned}$$

First of all, due to (3.13) it holds,

$$|||\hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1} - \Sigma_{\Lambda_j,\Lambda_j}^{-1}|||_{\mathsf{op}} \le \frac{\sigma_{\min}^{-2}\sqrt{s}\Delta_1}{1-\sigma_{\min}^{-1}\sqrt{s}\Delta_1} \le 2\sigma_{\min}^{-2}\sqrt{s}\Delta_1.$$

Since $A = \Sigma[\Theta^*]^\top$, we have

$$\begin{aligned}
\|A_{\Lambda_j,\cdot}(\mathbf{z}_j - \mathbf{z}_j^*)\| &\le \sigma_{\max} r_j \\
\|A_{\Lambda_j,\cdot}(\mathbf{z}_j + \mathbf{z}_j^*) - 2\lambda\mathbf{s}_j^*\| &\le \sigma_{\max}(2\|\mathbf{v}_j^*\| + r_j) + 2\lambda\sqrt{s}.
\end{aligned}$$

Then by Cauchy-Schwartz,

$$\begin{aligned}
|\tilde{\Phi}_\lambda(\mathscr{C}) - \bar{\Phi}_\lambda(\mathscr{C}) - \tilde{\Phi}_\lambda(\mathscr{C}^*) + \bar{\Phi}_\lambda(\mathscr{C}^*)| \le& \sigma_{\min}^{-2}\sqrt{s}\Delta_1\left(\sum_{j=1}^{K}\sigma_{\max} r_j\left\{\sigma_{\max}(2\|\mathbf{v}_j\| + r_j) + 2\lambda\sqrt{s}\right\}\right) \\
\le& 2\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}|||V^*|||_{\mathsf{F}}\Delta_1 r + 2\frac{\sigma_{\max}}{\sigma_{\min}^2}\lambda s\sqrt{K}\Delta_1 r \\
&+ \left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}\Delta_1 r^2.
\end{aligned}$$

Going further,

$$\Phi_\lambda(\mathscr{C}) - \tilde{\Phi}_\lambda(\mathscr{C}) = -\frac{1}{2}\sum_{j=1}^{K}\left((A_{\Lambda_j,\cdot} + \hat{A}_{\Lambda_j,\cdot})\mathbf{z}_j - 2\lambda\mathbf{s}_j^*\right)^\top \hat{\Sigma}_{\Lambda_j,\Lambda_j}^{-1}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j,$$

which implies that

$$
\begin{aligned}
&|\Phi_\lambda(\mathscr{C}) - \tilde{\Phi}_\lambda(\mathscr{C}) - \Phi_\lambda(\mathscr{C}^*) + \tilde{\Phi}_\lambda(\mathscr{C}^*)| \\
&\qquad \leq \frac{1}{2}\sum_{j=1}^{K}\left|\left((A_{\Lambda_j,\cdot} + \hat{A}_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\right)^\top \hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j\right| \\
&\qquad \frac{1}{2}\sum_{j=1}^{K}\left|\left((A_{\Lambda_j,\cdot} + \hat{A}_{\Lambda_j,\cdot})\mathbf{z}_j^* - 2\lambda\mathbf{s}_j^*\right)^\top \hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\right|
\end{aligned}
\tag{3.17}
$$

First notice, that due to Lemma 3.2 and (3.9) it holds,

$$
\begin{aligned}
\|(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\| &\leq \sqrt{s}\|\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot}\|_{\infty,\infty}\|\mathbf{z}_j - \mathbf{z}_j^*\|_1 \\
&\leq 1.55\sqrt{sn^*}\Delta_1 r_j^2.
\end{aligned}
$$

Therefore, it follows

$$
\|(\hat{A}_{\Lambda_j,\cdot} + A_{\Lambda_j,\cdot})(\mathbf{z}_j - \mathbf{z}_j^*)\| \leq 2\sigma_{\max} r_j + 1.55\sqrt{sn^*}\Delta_1 r_j^2.
$$

Moreover, using (3.10) we get

$$
\begin{aligned}
\|(\hat{A}_{\Lambda_j,\cdot} - A_{\Lambda_j,\cdot})\mathbf{z}_j\| &\leq \Delta_1 + 1.55\sqrt{sn^*}\Delta_1 r_j^2 \\
\|(\hat{A}_{\Lambda_j,\cdot} + A_{\Lambda_j,\cdot})\mathbf{z}_j^* - 2\lambda\mathbf{s}_j^*\| &\leq 2\sigma_{\max}\|\mathbf{v}_j\| + \Delta_1 + 2\lambda\sqrt{s}.
\end{aligned}
$$

and we also have $|||\hat{\Sigma}^{-1}_{\Lambda_j,\Lambda_j}|||_{\mathsf{op}} \leq 2\sigma_{\min}^{-1}$ due to the condition $\sigma_{\min}^{-1}\sqrt{s}\Delta_1 \leq 1/2$. Thus we get that the first sum of (3.17) is bounded by

$$
\begin{aligned}
&\sigma_{\min}^{-1}\sum_{j=1}^{K}\left(2\sigma_{\max} r_j + 1.55\sqrt{sn^*}\Delta_1 r_j^2\right)\left(\Delta_1 + 1.55\sqrt{sn^*}\Delta_1 r_j^2\right) \\
&\qquad \leq 2\frac{\sigma_{\max}}{\sigma_{\min}}\Delta_1\sqrt{K}r + 1.55\sigma_{\min}^{-1}\sqrt{sn^*}\Delta_1^2 r^2 + 3.1\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 r^3 + 2.5\sigma_{\min}^{-1} sn^*\Delta_1^2 r^4,
\end{aligned}
$$

while the second sum is bounded by

$$
\begin{aligned}
&\sigma_{\min}^{-1}\sum_{j=1}^{K}\left(2\sigma_{\max}\|\mathbf{v}_j^*\| + \Delta_1 + 2\lambda\sqrt{s}\right)\left(1.55\sqrt{sn^*}\Delta_1 r_j^2\right) \\
&\qquad \leq \frac{1.55}{\sigma_{\min}}\left(\sigma_{\max}\sqrt{sn^*} + \sqrt{sn^*}\Delta_1 + 2\lambda s\sqrt{n^*}\right)\Delta_1 r^2 \\
&\qquad \leq \frac{5}{\sigma_{\min}}\left(\sigma_{\max}\sqrt{sn^*} + \lambda s\sqrt{n^*}\right)\Delta_1 r^2
\end{aligned}
$$

where we used the fact that $\max_j \|\mathbf{v}_j^*\| \le |||V^*|||_{\mathsf{op}} = |||\Theta^*|||_{\mathsf{op}} < 1$ together with the condition $\Delta_1 \le \sigma_{\max}$. Combining all the bounds we get

$$
\begin{aligned}
&|\Phi_\lambda(\mathscr{C}) - \bar{\Phi}_\lambda(\mathscr{C}) - \Phi_\lambda(\mathscr{C}^*) + \bar{\Phi}_\lambda(\mathscr{C}^*)| \\
&\quad \le 2\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \sqrt{s}|||V^*|||_{\mathsf{F}} + 2\frac{\sigma_{\max}}{\sigma_{\min}^2}\lambda s\sqrt{K} + 2\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1 r \\
&\qquad + \left(5\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*} + 5\sigma_{\min}^{-1}\lambda s\sqrt{n^*} + 1.55\sigma_{\min}^{-1}\sqrt{sn^*}\Delta_1 + \left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\sqrt{s}\right)\Delta_1 r^2 \\
&\qquad + 3.1\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 r^3 \\
&\qquad + 2.5\sigma_{\min}^{-1} sn^*\Delta_1^2 r^4,
\end{aligned}
$$

where by $r \le 0.3$ and $\sqrt{sn^*}\Delta_1 \le \sigma_{\max}$ we can neglect the third and the fourth power, respectively, and thus the required bound follows. $\square$

Lemma 3.10. *There are numerical constant $c, C > 0$ such that the following holds. Suppose, the inequalities take place:*

$$
\sqrt{\frac{sn^* \log N}{T p_{\min}^2}} \le c\frac{a_0 \sigma_{\min}}{\sigma_{\max}^2}, \qquad n^* \ge \sigma_{\max}/\sigma_{\min}. \tag{3.18}
$$

Let $C\sigma_{\max}\sqrt{\frac{\log N}{T p_{\min}^2}} \le \lambda \le c\sigma_{\min}\tau_0 s^{-1}$, and set

$$
\bar{r} = 0.3 \wedge 0.18\sqrt{\alpha} \wedge 0.22\sqrt{\left(2\sigma_{\max}\alpha^{-1/2} + \sqrt{n^*}\Delta_1\right)^{-1}\lambda}.
$$

Then under the inequalities (3.9)–(3.13) *the clustering*

$$
\hat{\mathscr{C}} = \arg\min_{|||Z_{\mathscr{C}} - Z^*|||_{\mathsf{F}} \le r_{\max}} F_\lambda(\mathscr{C})
$$

satisfies

$$
|||Z_{\hat{\mathscr{C}}} - Z^*|||_{\mathsf{F}} \le \frac{C}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \lambda K\sqrt{s}.
$$

Proof. It is not hard to see that for $\Delta_1 = \sqrt{\frac{\log N}{T p_{\min}^2}}$ the inequalities required by Lemmas 3.7–3.9 are satisfied for $r \le \bar{r}$ due to (3.18) and conditions on λ and $\bar{r}$. Since obviously $\hat{\mathscr{C}}$ satisfies

$F_\lambda(\hat{\mathscr{C}}) \le F_\lambda(\mathscr{C}^*)$, we have for $r = |||Z_{\hat{\mathscr{C}}} - Z_{\mathscr{C}^*}|||_{\mathsf{F}} \le r_{\max}$

$$\begin{aligned}
F_\lambda(\hat{\mathscr{C}}) - F_\lambda(\mathscr{C}^*) \ge& \bar{\Phi}_\lambda(\mathscr{C}) - \bar{\Phi}_\lambda(\mathscr{C}) - |F_\lambda(\mathscr{C}) - \bar{\Phi}_\lambda(\mathscr{C}) - F_\lambda(\mathscr{C}^*) + \bar{\Phi}_\lambda(\mathscr{C}^*)| \\
\ge& \frac{a_0 r^2}{2}\left(1 - 10\alpha^{-1} r^2\right) - \lambda\sqrt{Ks}|||V^*|||_{\mathsf{F}} r \\
&- 4\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \sqrt{s}|||V^*|||_{\mathsf{F}} + \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1 r - 15\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 r^2 \\
=& \frac{a_0 r^2}{2}\left(1 - 10\alpha^{-1} r^2 - \frac{30}{a_0}\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1\right) \\
&- \lambda\sqrt{Ks}|||V^*|||_{\mathsf{F}} r - 4\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \sqrt{s}|||V^*|||_{\mathsf{F}} + \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1 r\,.
\end{aligned}$$

Since $\bar{r} \le 0.2\sqrt{\alpha}$ implies $10\alpha^{-1} r^2 \le \frac{1}{3}$, it holds by (3.18)

$$1 - 10\alpha^{-1} r^2 - \frac{30}{a_0}\frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{sn^*}\Delta_1 \ge \frac{1}{2}.$$

Therefore, after dividing by r, we get that such optimal clustering must satisfy

$$\frac{a_0}{4} r \le \lambda\sqrt{Ks}|||V^*|||_{\mathsf{F}} + 4\left(\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \sqrt{s}|||V^*|||_{\mathsf{F}} + \frac{\sigma_{\max}}{\sigma_{\min}}\sqrt{K}\right)\Delta_1.$$

Recalling that $|||V^*|||_{\mathsf{F}} \le \sqrt{K}$, $\Delta_1 = C\sigma_{\max}\sqrt{\frac{\log N}{T p_{\min}^2}}$ and $\Delta_2 = C\sqrt{\frac{s\log N}{T p_{\min}^2}}$ yields the result.

$\square$

Now we are ready to finalize the proof of Theorem 3.3. Firstly, we need to show that the clustering $\hat{\mathscr{C}}$ from the lemma above is locally optimal. By Lemma 3.5, any neighbouring to it clustering $\mathscr{C}'$ satisfies $|||Z_{\mathscr{C}'} - Z_{\hat{\mathscr{C}}}|||_{\mathsf{F}} \le \frac{2}{\sqrt{\alpha N/K}}$. Therefore,

$$|||Z_{\mathscr{C}'} - Z_{\mathscr{C}^*}|||_{\mathsf{F}} \le \frac{C}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \lambda K\sqrt{s} + 2\alpha^{-1/2}\sqrt{\frac{K}{N}},$$

and it is enough to check that this value is at most $\bar{r}$. We check that each of the terms is at most $\bar{r}/2$. For the first one it is enough to have,

$$\frac{C}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2 \alpha^{-1/2}\lambda K\sqrt{s} \le 0.09,$$
$$\frac{C^2}{a_0^2}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^4 \lambda\left(2\sigma_{\max}\alpha^{-1/2}+\sqrt{n^*}\Delta_1\right)K^2 s \le 0.012,$$

and both are satisfied due to the upper bound $\lambda \le c\kappa^{-4}(a_0^2/\sigma_{\max})K^{-2}s^{-1}$ and the requirement $\sqrt{\frac{sn^*\log N}{Tp_{\min}^2}} \le c$. For the second term we need

$$\alpha^{-1}\frac{K}{N} \le 0.008\alpha, \qquad \alpha^{-1}\left(2\sigma_{\max}\alpha^{-1/2}+\sqrt{n^*}\Delta_1\right)\frac{K}{N} \le \lambda,$$

both are satisfied once $N \ge C\alpha^2 K$ and $\lambda \ge C\sigma_{\max}\alpha^{-3/2}\frac{K}{N}$.

Moreover, by Lemma 3.7 we have for $\hat{\Theta} = Z_{\hat{\mathscr{C}}}\hat{V}_{\hat{\mathscr{C}},\lambda}$

$$\begin{aligned}|||\hat{\Theta}-\Theta^*|||_{\mathsf{F}} &\le |||Z_{\hat{\mathscr{C}}}(\hat{V}_{\hat{\mathscr{C}},\lambda}-V^*)^\top|||_{\mathsf{F}} + |||(Z_{\hat{\mathscr{C}}}-Z^*)V^*|||_{\mathsf{F}}\\ &\le 3\sigma_{\min}^{-1}\sqrt{Ks}\lambda + \frac{C}{a_0}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\gamma K\sqrt{s}\lambda,\end{aligned}$$

which finishes the proof.

3.6 Proof of Theorems 3.1 and 3.2

Recall that we have a time series,

$$Y_t = \sum_{k\ge 0}\Theta^k W_{t-k}, \qquad t \in \mathbb{Z}, \tag{3.19}$$

where $W_t \in \mathbb{R}^N$, $t \in \mathbb{Z}$ are independent vectors with $\mathsf{E}W_t = 0$ and $\mathrm{Var}(W_t) = S$. We also have that $|||\Theta|||_{\mathsf{op}} \le \gamma$ for some $\gamma < 1$ and the covariance $\Sigma = \mathrm{Var}(Y_t)$ reads as

$$\Sigma = \sum_{k\ge 0}\Theta^k S[\Theta^k]^\top.$$

We have the observations

$$Z_t = (\delta_{1t}Y_{1t}, \dots, \delta_{Nt}Y_{Nt})^\top, \qquad t = 1, \dots, T, \tag{3.20}$$

where $\delta_{it} \sim \mathrm{Be}(p_i)$ are independent Bernoulli random variables for each $i = 1, \dots, N$ and $t = 1, \dots, T$ and some $p_i \in (0, 1]$.

The proofs of both statements are based on a version of Bernstein matrix inequality presented in Chapter 4, Proposition 4.3.

Theorem 3.4 (Klochkov and Zhivotovskiy (2018), Proposition 4.1)**.** *Suppose, the matrices A_t for $t = 1, \dots, T$ are independent and let $M = \max_t \left\| \|\|A_t\|\|_{\mathsf{op}} \right\|_{\psi_1}$ is finite. Then, $S_T = \sum_{t=1}^T A_t$ satisfies for any $u \geq 1$*

$$\mathsf{P}\left(\|\|S_T - \mathsf{E}S_T\|\|_{\mathsf{op}} > C\left(\sqrt{\sigma^2(\log N + u)} + M \log T (\log N + u) \right) \right) \leq e^{-u},$$

where $\sigma^2 = \|\| \sum_{t=1}^T \mathsf{E}A_t^\top A_t \|\|_{\mathsf{op}} \vee \|\| \sum_{t=1}^T \mathsf{E}A_t A_t^\top \|\|_{\mathsf{op}}$ and C is an absolute constant.

Let $\delta_t = (\delta_{t1}, \dots, \delta_{tN})^\top$ denotes the vector with Bernoilli variables from above corresponding to the time point t. In what follows we consider the following matrices,

$$A_{t,t'}^{k,j} = \operatorname{diag}\{\delta_t\} \Theta^k W_{t-k} W_{t'-j}^\top [\Theta^j]^\top \operatorname{diag}\{\delta_{t'}\},$$

so that since $Z_t = \sum_{k \geq 0} \operatorname{diag}\{\delta_t\} \Theta^k W_{t-k}$, we have

$$Z_t Z_t^\top = \sum_{k,j \geq 0} \operatorname{diag}\{\delta_t\} \Theta^k W_{t-k} W_{t-j}^\top [\Theta^j]^\top \operatorname{diag}\{\delta_t\} = \sum_{k,j \geq 0} A_{t,t}^{k,j}.$$

Therefore, the decomposition takes place

$$\Sigma^* = \sum_{k,j \geq 0} S_{k,j}, \qquad S_{k,j} = \frac{1}{T} \sum_{t=1}^T A_{t,t}^{k,j}, \tag{3.21}$$

and we shall analyze the sum for each pair of $k, j \geq 0$ separately. We first introduce two technical lemmas. In what follows we assume w.l.o.g. that $\|\|S\|\|_{\mathsf{op}} = 1$, since if we scale it, all the covariances and estimators scale correspondingly.

Lemma 3.11. *Under the assumptions of Proposition 3.1 it holds,*

$$\| \| P \operatorname{diag}\{\mathbf{p}\}^{-1} \operatorname{Diag}(A^{k,j}_{t,t'}) Q \|_{\mathsf{op}} \|_{\psi_1} \le C p_{\min}^{-1} \sqrt{M_1 M_2} \gamma^{k+j},$$
$$\| \| P \operatorname{diag}\{\mathbf{p}\}^{-1} \operatorname{Off}(A^{k,j}_{t,t'}) \operatorname{diag}\{\mathbf{p}\}^{-1} Q \|_{\mathsf{op}} \|_{\psi_1} \le C p_{\min}^{-2} \sqrt{M_1 M_2} \gamma^{k+j},$$

with some $C = C(L) > 0$.

Proof. Denote for simplicity $\mathbf{x} = \Theta^k W_{t-k}$, $\mathbf{y} = \Theta^j W_{t'-j}$, as well as $\mathbf{x}^\delta = \operatorname{diag}\{\delta_t\}\mathbf{x}$, $\mathbf{y}^\delta = \operatorname{diag}\{\delta_t\}\mathbf{y}$, such that $A^{k,j}_{t,t'} = \mathbf{x}^\delta [\mathbf{y}^\delta]^\top$. Since W_t are subgaussian and $\|\Theta^k S \Theta^k\|_{\mathsf{op}} \le \gamma^{2k}$, we have for each $\mathbf{u} \in \mathbb{R}^N$ that

$$\log \mathsf{E} \exp(\mathbf{u}^\top \mathbf{x}) \le C' \gamma^{2k} \|\mathbf{u}\|^2, \tag{3.22}$$

and since δ_t takes values in $[0,1]^N$, same takes place for $\mathbf{x}^\delta$. By Theorem 2.1 in Hsu et al. (2012) it holds for any matrix A and vector $\mathbf{u} \in \mathbb{R}^N$,

$$\| \| A \mathbf{x}^\delta \| \|_{\psi_2} \le C'' \gamma^k \|A\|_{\mathsf{F}}, \qquad \|\mathbf{u}^\top \mathbf{x}^\delta\|_{\psi_2} \le C'' \gamma^k \|\mathbf{u}\|, \tag{3.23}$$

and, similarly,

$$\| \| A \mathbf{y}^\delta \| \|_{\psi_2} \le C'' \gamma^j \|A\|_{\mathsf{F}}, \qquad \|\mathbf{u}^\top \mathbf{y}^\delta\|_{\psi_2} \le C'' \gamma^j \|\mathbf{u}\|.$$

We first deal with the diagonal term. Let $P = \sum_{i=1}^{M_1} \mathbf{u}_j \mathbf{u}_j^\top$ be its eigen-decomposition with $\|\mathbf{u}_j\| = 1$, then

$$\begin{aligned}
\| \| P \operatorname{diag}(\mathbf{x}^\delta) \|_{\mathsf{op}} \|_{\psi_2}^2 &= \| \| \operatorname{diag}(\mathbf{x}^\delta) P \operatorname{diag}(\mathbf{x}^\delta) \|_{\mathsf{op}} \|_{\psi_1} \le \sum_{j=1}^{M_1} \| \| \operatorname{diag}(\mathbf{x}_\delta) \mathbf{u}_j \mathbf{u}_j^\top \operatorname{diag}(\mathbf{x}^\delta) \|_{\mathsf{op}} \|_{\psi_1} \\
&= \sum_{j=1}^{M_1} \| \| \operatorname{diag}(\mathbf{u}_j) \mathbf{x}^\delta \| \|_{\psi_2}^2,
\end{aligned}$$

where each term in the latter is bounded by γ^{2k} due the fact that $\| \operatorname{diag}(\mathbf{u}_j) \|_{\mathsf{F}} = 1$. Summing up and taking square root we arrive at $\big\| \| P \operatorname{diag}(\mathbf{x}^\delta) \|_{\mathsf{op}} \big\|_{\psi_2} \le \sqrt{C'' M_1} \gamma^k$. Taking into account similar bound for $Q \operatorname{diag}(\mathbf{y}^\delta)$, we have by Hölder inequality

$$\begin{aligned}
\| \| P \operatorname{diag}\{\delta\}^{-1} \operatorname{diag}(\mathbf{x}^\delta) \operatorname{diag}(\mathbf{y}^\delta) Q \|_{\mathsf{op}} \|_{\psi_1} \le & p_{\min}^{-1} \| \| P \operatorname{diag}(\mathbf{x}^\delta) \|_{\mathsf{op}} \big\|_{\psi_2} \| \| Q \operatorname{diag}(\mathbf{y}^\delta) \|_{\mathsf{op}} \|_{\psi_2} \\
\le & C'' \sqrt{M_1 M_2} \gamma^{k+j},
\end{aligned}$$

which yields the bound for the diagonal. As for the off-diagonal, consider first the whole matrix,

$$\|\|\|P\mathbf{x}^\delta[\mathbf{y}^\delta]^\top Q\|\|_{\mathsf{op}}\|_{\psi_1} \leq \|\|\|P\mathbf{x}^\delta\|\|\|_{\psi_2}\|\|\|Q\mathbf{y}^\delta\|\|\|_{\psi_2} \leq (C'')^2\sqrt{M_1M_2}\gamma^{j+k},$$

and since $\mathrm{Off}(A_{t,t'}^{j,k}) = A_{t,t'}^{j,k} - \mathrm{Diag}(A_{t,t'}^{j,k})$, the bound follows from the triangular inequality. □

The following technical lemma will help us to upper-bound σ^2 in Theorem 3.4.

Lemma 3.12. *Let $\delta_1,\ldots,\delta_N$ consists of independent Bernoilli components with probabilities of success $p_1,\ldots,p_N$ and set $p_{\min} = \min_{i\leq N} p_i$. Let $\mathbf{a},\mathbf{b} \in \mathbb{R}^N$ be two arbitrary vectors. It holds,*

$$\begin{aligned}
\mathsf{E}\left(\sum_i \frac{\delta_i}{p_i} a_i b_i\right)^2 &\leq p_{\min}^{-1}\|\mathbf{a}\|^2\|\mathbf{b}\|^2, \\
\mathsf{E}\left(\sum_{i\neq j} \frac{\delta_i\delta_j}{p_ip_j} a_i b_j\right)^2 &\leq 32 p_{\min}^{-2}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 + 4\left(\sum_i a_i\right)^2\left(\sum_i b_i\right)^2.
\end{aligned}$$

Additionally, if $\delta_1',\ldots,\delta_N'$ are independent copies of $\delta_1,\ldots,\delta_N$, it holds

$$\mathsf{E}\left(\sum_{i,j} \frac{\delta_i\delta_j'}{p_ip_j} a_i b_j\right)^2 \leq 4p_{\min}^{-2}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 + 4\left(\sum_i a_i\right)^2\left(\sum_i b_i\right)^2.$$

Proof. It holds,

$$\begin{aligned}
\mathsf{E}\left(\sum_i \frac{\delta_i}{p_i} a_i b_i\right)^2 &= \sum_{i,j} \mathsf{E}\frac{\delta_i\delta_j}{p_ip_j} a_i b_i a_j b_j = \sum_{i,j}\{1 + \mathbf{I}(i=j)(p_i^{-1}-1)\} a_i b_i a_j b_j \\
&\leq \left(\sum_i a_i b_i\right)^2 + (p_{\min}^{-1}-1)\sum_i a_i^2 b_i^2 \\
&\leq \|\mathbf{a}\|^2\|\mathbf{b}\|^2 + (p_{\min}^{-1}-1)\|\mathbf{a}\|^2\|\mathbf{b}\|^2.
\end{aligned}$$

To show the second inequality we use decoupling (Theorem 6.1.1 in Vershynin (2018)) and the trivial inequality $(x+y)^2 \leq 2x^2+2y^2$,

$$\begin{aligned}\mathsf{E}\left(\sum_{i\neq j}\frac{\delta_i\delta_j}{p_ip_j}a_ib_j\right)^2 \leq& 2\left(\sum_{i\neq j}a_ib_j\right)^2+2\mathsf{E}\left(\sum_{i\neq j}\frac{(\delta_i-p_i)(\delta_j-p_j)}{p_ip_j}a_ib_j\right)^2\\ \leq& 2\left(\sum_{i\neq j}a_ib_j\right)^2+32\mathsf{E}\left(\sum_{i\neq j}\frac{(\delta_i-p_i)(\delta_j'-p_j)}{p_ip_j}a_ib_j\right)^2.\end{aligned} \tag{3.24}$$

Denote for simplicity $\overline{\delta}_i=\delta_i-p_i$ and $\overline{\delta}_i'=\delta_i'-p_i$. Since the latter are centred we have,

$$\mathsf{E}\left(\sum_{i\neq j}\frac{\overline{\delta}_i\overline{\delta}_j'}{p_ip_j}a_ib_j\right)^2=\sum_{\substack{i\neq j\\ k\neq l}}\frac{\mathsf{E}\overline{\delta}_i\overline{\delta}_k}{p_ip_k}\frac{\mathsf{E}\overline{\delta}_j'\overline{\delta}_l'}{p_jp_j}a_ia_kb_jb_l \tag{3.25}$$

note that the expectation $\mathsf{E}\overline{\delta}_i\overline{\delta}_k$ is only non-vanishing when $i=k$, in which case it holds $\mathsf{E}\overline{\delta}_i^2=p_i-p_i^2$. Taking into account similar property of $\mathsf{E}\overline{\delta}_j'\overline{\delta}_l'$ we have that the sum above is equal to

$$\sum_{i\neq j}\frac{(p_i-p_i^2)(p_j-p_j^2)}{p_i^2p_j^2}a_i^2b_j^2\leq(p_{\min}^{-1}-1)^2\sum_{i,j}a_i^2b_j^2\leq(p_{\min}^{-1}-1)^2\|\mathbf{a}\|^2\|\mathbf{b}\|^2.$$

It is left to notice that

$$\left(\sum_{i\neq j}a_ib_j\right)^2\leq 2\left(\sum_{i,j}a_ib_j\right)^2+2\left(\sum_{i}a_ib_j\right)^2\leq 2\left(\sum_i a_i\right)^2\left(\sum_i b_i\right)^2+2\|\mathbf{a}\|^2\|\mathbf{b}\|^2,$$

which recalling (3.24) and noting that $32(p_{\min}^{-1}-1)^2+4\leq 32p_{\min}^{-2}$ for $p_{\min}\in[0,1]$, completes the proof.

Similarly to (3.25) we can show the third inequality. □

Now we apply Bernstein matrix inequality to the sum S_{kj} defined in (3.21), dealing separately with diagonal and off-diagonal parts. After that we present the proof of Proposition 3.1.

Lemma 3.13. *Under the assumptions of Proposition 3.1 for each $u \ge 1$ it holds with probability at least $1 - e^{-u}$*

$$
\begin{aligned}
&|||P \operatorname{diag}\{\mathbf{p}\}^{-1}(\operatorname{Diag}(S_{k,j}) - \mathsf{E}\operatorname{Diag}(S_{k,j}))Q|||_{\mathsf{op}} \\
&\qquad \le C\gamma^{k+j}\left(\sqrt{\frac{M_1 \vee M_2(\log N + u)}{T p_{\min}}} \bigvee \frac{\sqrt{M_1 M_2}(\log N + u)}{T p_{\min}}\right)
\end{aligned}
$$

where $C = C(K)$ only depends on K.

Proof. Note that,

$$
P \operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Diag}(S_{kj})Q = T^{-1}\sum_{t=1}^{T} A_t, \qquad A_t = P \operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Diag}(A_{t,t}^{k,j})Q.
$$

By Lemma 3.11 we have $\| |||A_t|||_{\mathsf{op}} \|_{\psi_1} \le C p_{\min}^{-1}\sqrt{M_1 M_2}\gamma^{k+j}$. Moreover, using decomposition $Q = \sum_{j=1}^{M_2} \mathbf{u}_j \mathbf{u}_j$, we have

$$
\begin{aligned}
|||\mathsf{E} A_t A_t^\top|||_{\mathsf{op}} &\le |||\mathsf{E}\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Diag}(A_{t,t}^{k,j})Q\operatorname{Diag}(A_{t,t}^{k,j})\operatorname{diag}\{\mathbf{p}\}^{-1}|||_{\mathsf{op}} \\
&\le \sum_{j=1}^{M_2} |||\mathsf{E}\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Diag}(A_{t,t}^{k,j})\mathbf{u}_j\mathbf{u}_j^\top\operatorname{Diag}(A_{t,t}^{k,j})\operatorname{diag}\{\mathbf{p}\}^{-1}|||_{\mathsf{op}} \\
&\le \sum_{j=1}^{M_2} \sup_{\|\gamma\|=1} \mathsf{E}(\gamma^\top \operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Diag}(A_{t,t}^{k,j})\mathbf{u}_j)^2
\end{aligned}
$$

By definition, $\operatorname{Diag}(A_{t,t}^{k,j}) = \operatorname{diag}\{\delta_{ti}x_i y_i\}_{i=1}^N$ for $\mathbf{x} = \Theta^k W_{t-k}$, $\mathbf{y} = \Theta^j W_{t-j}$. Let E_δ denotes the expectation w.r.t. the Bernoulli variables and conditioned on everything else. Setting $\mathbf{a} = (x_1\gamma_1, \dots, x_N\gamma_N)^\top)$ and $\mathbf{b} = (y_1 u_1, \dots, y_N u_N)^\top$, we have by the first inequality of Lemma 3.12,

$$
\begin{aligned}
\mathsf{E}(\gamma^\top \operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Diag}(A_{t,t}^{k,j})\mathbf{u}_j)^2 &= \mathsf{E}\mathsf{E}_\delta\left(\sum_i \gamma_i x_i \frac{\delta_{ti}}{p_i} y_i u_i\right)^2 \\
&\le p_{\min}^{-1}\mathsf{E}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 \\
&\le p_{\min}^{-1}\mathsf{E}^{1/2}\|\mathbf{a}\|^4\mathsf{E}^{1/4}\|\mathbf{b}\|^4.
\end{aligned}
$$

Observe that,

$$
\|\mathbf{a}\|^2 = \sum_i \gamma_i^2 x_i^2 = \mathbf{x}^\top \operatorname{diag}\{\gamma\}^2 \mathbf{x},
$$

so since $\mathrm{tr}(\mathrm{diag}\{\gamma\}^2) = 1$ and due to (3.22) and by Theorem 2.1 Hsu et al. (2012) it holds $\mathsf{E}^{1/2}\|\mathbf{a}\|^4 \le \|\|\mathbf{a}\|^2\|_{\psi_1} \le C'\gamma^{2k}$. Similarly, it holds $\mathsf{E}^{1/2}\|\mathbf{a}\|^4 \le C'\gamma^{2j}$, which together implies

$$|||\mathsf{E}A_tA_t^\top|||_{\mathsf{op}} \vee |||\mathsf{E}A_t^\top A_t^\top|||_{\mathsf{op}} \le C'' M_2 \vee M_1 \gamma^{2k+2j}.$$

Now notice that A_t is not necessary an independent sequence, as A_t depends directly on $(W_{t-k}, W_{t-j}, \delta_t)$, which might intersect with e.g. $t' = t + |j-k|$. However, if we take a set $I \subset [1,T]$ such that any two $t, t' \in I$ satisfy $|t'-t| \neq |j-k|$ then obviously the sequence $(A_t)_{t\in I}$ is independent. We separate the whole interval $[1,T]$ into two such independent sets,

$$\begin{aligned} I_1 =& \{t \in [1,T] : \lceil t/|j-k| \rceil \text{ is odd } \}, \\ I_2 =& \{t \in [1,T] : \lceil t/|j-k| \rceil \text{ is even } \} \\ =& [1,T] \setminus I_1. \end{aligned} \tag{3.26}$$

Indeed, if for $t, t' \in I_1$ then $\lceil t/|j-k| \rceil$ and $\lceil t'/|j-k| \rceil$ are either equal or differ in at least two, so that in the first case we have $|t-t'| < |j-k|$ and in the second $|t-t'| > |j-k|$. Since both intervals have, very roughly, at most T elements, it holds by Theorem 3.4 with probability at least $1-e^{-u}$ for both j,

$$\begin{aligned} &|||\sum_{t\in I_j} A_t - \mathsf{E}A_t|||_{\mathsf{op}} \\ &\quad \le C\gamma^{j+k}\left(\sqrt{p_{\min}^{-1}(M_1 \vee M_2)T(\log N + u)} \vee p_{\min}^{-1}\sqrt{M_1M_2}(\log N + u)\log T\right), \end{aligned}$$

so summing up the two and dividing by T we get the result. □

Lemma 3.14. *Under the assumptions of Proposition 3.1 for each $u \ge 1$ it holds with probability at least $1-e^{-u}$*

$$\begin{aligned} &|||P\,\mathrm{diag}\{\mathbf{p}\}^{-1}(\mathrm{Off}(S_{k,j}) - \mathsf{E}\,\mathrm{Off}(S_{k,j}))\,\mathrm{diag}\{\mathbf{p}\}^{-1}Q|||_{\mathsf{op}} \\ &\qquad \le C\gamma^{k+j}\left(\sqrt{\frac{M_1 \vee M_2(\log N + u)}{Tp_{\min}^2}} \bigvee \frac{\sqrt{M_1M_2}(\log N + u)\log T}{Tp_{\min}^2}\right) \end{aligned}$$

where $C = C(K)$ only depends on K.

Proof. It holds,

$$P\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Off}(S_{kj})\operatorname{diag}\{\mathbf{p}\}^{-1}Q = T^{-1}\sum_{t=1}^{T} B_t,\;\; B_t = P\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Off}(A^{k,j}_{t,t})\operatorname{diag}\{\mathbf{p}\}^{-1}Q.$$

By Lemma 3.11 we have $\| \|\|B_t\|\|_{\mathsf{op}}\|_{\psi_1} \le Cp_{\min}^{-2}\sqrt{M_1M_2}\gamma^{k+j}$. Using decomposition $Q = \sum_{j=1}^{M_2}\mathbf{u}_j\mathbf{u}_j$ with $\|\mathbf{u}_j\| = 1$ we get that

$$\begin{aligned}
\|\|\mathsf{E}B_tB_t^\top\|\|_{\mathsf{op}} \le& \|\|\mathsf{E}\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Off}(A^{k,j}_{t,t})\operatorname{diag}\{\mathbf{p}\}^{-1}Q\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Off}(A^{k,j}_{t,t})\operatorname{diag}\{\mathbf{p}\}^{-1}\|\|_{\mathsf{op}}\\
\le& \sum_{j=1}^{M_2}\|\|\mathsf{E}\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Off}(A^{k,j}_{t,t})\operatorname{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j\mathbf{u}_j^\top\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Off}(A^{k,j}_{t,t})\operatorname{diag}\{\mathbf{p}\}^{-1}\|\|_{\mathsf{op}}\\
\le& \sum_{j=1}^{M_2}\sup_{\|\gamma\|=1}\mathsf{E}(\gamma^\top\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Off}(A^{k,j}_{t,t})\operatorname{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j)^2
\end{aligned}$$

Again, using the notation $\mathbf{x} = \Theta^kW_{t-k}$, $\mathbf{y} = \Theta^jW_{t-j}$ and $\mathbf{a} = \operatorname{diag}\{\gamma\}\mathbf{x}$, $\mathbf{b} = \operatorname{diag}\{\mathbf{u}\}\mathbf{y}$, we have that $\operatorname{Off}(A^{j,k}_{t,t}) = \operatorname{Off}(\mathbf{x}\mathbf{y}^\top)$, therefore by Lemma 3.12

$$\begin{aligned}
\mathsf{E}(\gamma^\top\operatorname{diag}\{\mathbf{p}\}^{-1}\operatorname{Off}(A^{k,j}_{t,t})\operatorname{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j)^2 =& \mathsf{E}\mathsf{E}_\delta\left(\sum_{i\neq j}\gamma_i\frac{\delta_{it}}{p_i}x_iy_j\frac{\delta_{jt}}{\delta_j}u_j\right)^2\\
=& \mathsf{E}\mathsf{E}_\delta\left(\sum_{i\neq j}\frac{\delta_{it}}{p_i}\frac{\delta_{jt}}{\delta_j}a_ib_j\right)^2\\
\le& 32p_{\min}^{-2}\mathsf{E}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 + 4\mathsf{E}\left(\sum_i a_i\right)^2\left(\sum_i b_i\right)^2.
\end{aligned}$$

From the proof of Lemma 3.14 we know that $\mathsf{E}\|\mathbf{a}\|^2\|\mathbf{b}\|^2 \le C'\gamma^{2k+2j}$. Moreover, we have $\sum_i a_i = \gamma^\top\mathbf{x}$ and $\sum_i b_i = \mathbf{u}^\top\mathbf{y}$. Thus, by (3.23) it holds $\mathsf{E}^{1/4}\|\gamma^\top\mathbf{x}\|^4 \le \|\gamma^\top\mathbf{x}\|_{\psi_2} \le C'\gamma^j$ and, similarly, $\mathsf{E}^{1/4}\|\mathbf{u}^\top\mathbf{y}\|^4 \le C'\gamma^k$. Putting those bounds together and applying Cauchy-Schwarz inequality, we have

$$\|\|\mathsf{E}B_tB_t^\top\|\|_{\mathsf{op}} \le C''p_{\min}^{-2}M_2\gamma^{2k+2j}.$$

By analogy, we have

$$\|\|\mathsf{E}B_tB_t^\top\|\|_{\mathsf{op}} \vee \|\|\mathsf{E}B_t^\top B_t\|\|_{\mathsf{op}} \le C''p_{\min}^{-2}M_1 \vee M_2\gamma^{2k+2j}.$$

Applying the same sample splitting (3.26) we obtain the bound

$$\left\|\left|\sum_t A_t - \mathsf{E}A_t\right|\right\|_{\mathsf{op}} \le C\gamma^{j+k}\left(\sqrt{p_{\min}^{-2}(M_1 \vee M_2)T(\log N + u)} \vee p_{\min}^{-2}\sqrt{M_1M_2}(\log N + u)\right),$$

which divided by T provides the result. $\square$

Proof of Theorem 3.1. Set,

$$S_{k,j}^{\delta} = \mathrm{diag}\{\mathbf{p}\}^{-1}\,\mathrm{Diag}(S_{k,j}) - \mathrm{diag}\{\delta\}^{-1}\,\mathrm{Off}(S_{k,j})\,\mathrm{diag}\{\delta\}^{-1},$$

so that by the union of bounds in Lemmas 3.14, 3.13 for each $u \ge 1$

$$\||P(S_{k,j}^{\delta} - \mathsf{E}S_{k,j}^{\delta})Q\||_{\mathsf{op}} > C\gamma^{k+j}\left(\sqrt{\frac{M_1 \vee M_2(\log N + u)}{Tp_{\min}^2}} \bigvee \frac{\sqrt{M_1M_2}(\log N + u)}{Tp_{\min}^2}\right)$$

holds with probability at least $1 - e^{-u}$. Take a union of those bounds for each k, j with $u = u_{k,j} = k + j + 1 + u'$. The total probability of complementary event is at most

$$\sum_{k,j\ge 0} e^{-k-j-1-u} = e^{-1-u}\left(\sum_{k\ge 0} e^{-k}\right)^2 = e^{-u}/(e-1) < e^{-u}.$$

On such event it holds

$$\begin{aligned}
\||P(\hat{\Sigma} - \mathsf{E}\Sigma)Q\||_{\mathsf{op}} \le & \sum_{k,j\ge 0} \||P(S_{k,j}^{\delta} - \mathsf{E}S_{k,j}^{\delta})Q\||_{\mathsf{op}} \\
\le & C\sum_{k,j\ge 0}\gamma^{k+j}\left(\sqrt{\frac{M_1 \vee M_2(\log N + u_{k,j})}{Tp_{\min}^2}} \bigvee \frac{\sqrt{M_1M_2}(\log N + u_{k,j})}{Tp_{\min}^2}\right) \\
\le & C'\left[\sum_{k,j\ge 0}\gamma^{k+j}\right]\left(\sqrt{\frac{(M_1 \vee M_2)\log N}{Tp_{\min}^2}} \bigvee \frac{\sqrt{M_1M_2}\log N}{Tp_{\min}^2}\right) \\
& + C\left[\sum_{k,j}(k+j)\gamma^{k+j}\right]\left(\sqrt{\frac{(M_1 \vee M_2)u}{Tp_{\min}^2}} \bigvee \frac{\sqrt{M_1M_2}u}{Tp_{\min}^2}\right),
\end{aligned}$$

which completes the proof due to the equalities

$$\sum_{k,j\geq 0}\gamma^{k+j} = \left(\sum_{k\geq 0}\gamma^k\right)^2 = \frac{1}{(1-\gamma)^2}$$
$$\sum_{k,j\geq 0}(k+j)\gamma^{k+j} = 2\sum_{k,j\geq 0}k\gamma^{k+j} = \frac{2}{(1-\gamma)}\sum_{k\geq 0}k\gamma^k = \frac{2}{(1-\gamma)^3}.$$

□

Proof of Theorem 3.2. Recall the definition,

$$A_{t,t'}^{k,j} = \operatorname{diag}\{\delta_t\}\Theta^k W_{t-k}W_{t'-j}^\top[\Theta^j]^\top \operatorname{diag}\{\delta_{t'}\}.$$

Then, it holds

$$Z_t Z_{t+1}^\top = \sum_{k,j\geq 0}\operatorname{diag}\{\delta_t\}\Theta^k W_{t-k}W_{t+1-j}^\top[\Theta^j]^\top \operatorname{diag}\{\delta_{t+1}\} = \sum_{k,j\geq 0}A_{t,t+1}^{k,j},$$

and the decomposition takes place,

$$A^* = \sum_{k,j\geq 0}S_{k,j}, \qquad S_{k,j} = \frac{1}{T-1}\sum_{t=1}^{T-1}A_{t,t+1}^{k,j}.$$

We first apply Bernstein matrix for each $S_{k,j}$ separately. Observe that

$$P\operatorname{diag}\{\mathbf{p}\}^{-1}S_{k,j}\operatorname{diag}\{\mathbf{p}\}^{-1}Q = \frac{1}{T-1}\sum_{t=1}^{T-1}B_t, \qquad B_t = P\operatorname{diag}\{\mathbf{p}\}^{-1}A_{t,t+1}^{k,j}\operatorname{diag}\{\mathbf{p}\}^{-1}Q.$$

By Lemma 3.11 each term satisfies,

$$\max_t \|\,|||B_t|||_{\mathsf{op}}\|_{\psi_1} \leq C\sqrt{M_1M_2}\gamma^{k+j}.$$

Furthermore, let $Q = \sum_{j=1}^{M_2}\mathbf{u}_j\mathbf{u}_j^\top$ with unit vectors $\mathbf{u}_j$. Also, denoting $\mathbf{x} = \Theta^k W_{t-k}$ and $\mathbf{y} = \Theta^k W_{t+1-k}$ it holds $A_{t,t+1}^{k,j} = \operatorname{diag}\{\delta_t\}\mathbf{x}\mathbf{y}^\top\operatorname{diag}\{\delta_{t+1}\}$. Then, we have for each unit

$\gamma \in \mathbb{R}^N$ and using Lemma 3.12,

$$\begin{aligned}
&\mathsf{E}(\gamma^\top \operatorname{diag}\{\mathbf{p}\}^{-1} A^{k,j}_{t,t+1} \operatorname{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j)^2 \\
&= \mathsf{E}\mathsf{E}_\delta \left(\sum_{i,j} \gamma_i x_i \frac{\delta_{ti}}{p_i} \frac{\delta_{t+1,j}}{p_j} y_j u_j \right)^2 \\
&\le p_{\min}^{-2} \mathsf{E}\|\operatorname{diag}\{\gamma\}\mathbf{x}\|^2 \|\operatorname{diag}\{\mathbf{u}\}\mathbf{y}\|^2 + \mathsf{E}(\gamma^\top \mathbf{x})(\mathbf{u}^\top \mathbf{y})^2,
\end{aligned}$$

which due to the subgaussianity of $\mathbf{x}$ and $\mathbf{y}$ yields,

$$\begin{aligned}
\mathsf{E}\|\operatorname{diag}\{\gamma\}\mathbf{x}\|^2 \|\operatorname{diag}\{\mathbf{u}\}\mathbf{y}\|^2 &\le \mathsf{E}^{1/2}\|\operatorname{diag}\{\gamma\}\mathbf{x}\|^4 \mathsf{E}^{1/2}\|\operatorname{diag}\{\mathbf{u}\}\mathbf{y}\|^4 \\
&\le C' \gamma^{2k+2j} \\
\mathsf{E}(\gamma^\top \mathbf{x})(\mathbf{u}^\top \mathbf{y})^2 &\le \mathsf{E}^{1/2}(\gamma^\top \mathbf{x})^4 \mathsf{E}^{1/2}(\mathbf{u}^\top \mathbf{y})^4 \\
&\le C' \gamma^{2k+2j}.
\end{aligned}$$

Therefore, we get that

$$|||\mathsf{E} B_t B_t^\top|||_{\mathsf{op}} = \sup_{\|\gamma\|=1} \sum_{j=1}^{M_2} \mathsf{E}\left(\gamma^\top \operatorname{diag}\{\mathbf{p}\}^{-1} A^{k,j}_{t,t+1} \operatorname{diag}\{\mathbf{p}\}^{-1}\mathbf{u}_j\right)^2 \le C'' p_{\min}^{-2} M_2 \gamma^{2k+2j}.$$

Taking similar derivations we can arrive at

$$\sigma^2 = |||\mathsf{E} B_t B_t^\top|||_{\mathsf{op}} \vee |||\mathsf{E} B_t^\top B_t|||_{\mathsf{op}} \le C'' p_{\min}^{-2} (M_1 \vee M_2) \gamma^{2k+2j}.$$

Now we separate the indices $t = 1, \dots, T$ into four subsets, such that each corresponds to a set of independent matrices B_t. Since each B_t is generated by $(W_{t-k}, W_{t+1-j}, \delta_t)$, and δ_{t+1}, we simply need to ensure that none of the pair of indices t, t' from the same subset satisfies $|t - t'| = |k - j + 1|$ nor $|t - t'| = 1$. This can be satisfied by the following separation. First, we separate the indices into two subsets with odd and even indices, respectively, so that none of the subsets contains two indices with $|t - t'| = 1$. Then, both of the subsets need to be separated into two others according to the scheme (3.26), so that the assertion $|t - t'| = |k - j + 1|$ is avoided within each subset. Therefore, applying Bernstein inequality, Theorem 3.4, to each sum separately and then summing up, we get that for each $u \ge 1$ with probability at least $1 - e^{-u}$,

$$\begin{aligned}
&\|\!|P \operatorname{diag}\{\delta\}^{-1}(S_{k,j} - \mathsf{E} S_{k,j}) \operatorname{diag}\{\delta\}^{-1} Q\|\!|_{\mathsf{op}} \\
&\qquad \le C \left(\sqrt{p_{\min}^{-2}(M_1 \vee M_2) T (\log N + u)} \bigvee \sqrt{M_1 M_2} (\log N + u) \log T \right).
\end{aligned}$$

Similarly to the proof of Proposition 3.1 we take the union of those bounds for each i, j with $u = j + k + u'$ and then the result follows. $\square$

Chapter 4

Uniform Hanson-Wright inequality with subgaussian entries

The concentration properties of quadratic forms of random variables is a classic topic in probability. The well-known result is due to Hanson and Wright (we refer to the form of this inequality presented in Rudelson and Vershynin (2013)) which claims that if A is an $n \times n$ real matrix and $X = (X_1, \dots, X_n)$ is a random vector in $\mathbb{R}^n$ with independent centered coordinates satisfying $\max_i \|X_i\|_{\psi_2} \le K$ (we will recall the definition of $\|\cdot\|_{\psi_2}$ below) then for all $t \ge 0$

$$\mathsf{P}(|X^\top AX - \mathsf{E}X^\top AX| \ge t) \le 2\exp\left(-c\min\left\{\frac{t^2}{K^4\|A\|_{\mathrm{HS}}^2}, \frac{t}{K^2\|A\|}\right\}\right), \tag{4.1}$$

for some absolute $c > 0$ and $\|A\|_{\mathrm{HS}} = \sqrt{\sum_{i,j} A_{i,j}^2}$ defines the Hilbert-Schmidt norm and $\|A\|$ is an operator norm of A. An important extension of these results is when instead of just one matrix A we have a family of matrices $\mathscr{A}$ and want to understand the behaviour of random quadratic forms simultaneously for all matrices in the family. As a concrete example we consider an order-2 Rademacher chaos: given a family $\mathscr{A} \subset \mathbb{R}^{n\times n}$ of $n \times n$ real symmetric matrices with zero diagonal, that is for all $A \in \mathscr{A}$ we have $A_{ii} = 0$ for all $i = 1, \dots, n$, one wants to study the following random variable

$$Z = \sup_{A\in\mathscr{A}} \sum_{i,j=1}^{n} A_{ij}\varepsilon_i\varepsilon_j = \sup_{A\in\mathscr{A}} \varepsilon^\top A\varepsilon,$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^\top$ is a sequence of independent Rademacher signs, taking values ± 1 with equal probabilities. In the celebrated paper Talagrand (1996) it was shown, in particular, that there is an absolute constant $c > 0$, such that for any $t \geq 0$

$$\mathsf{P}(|Z - \mathsf{E}Z| \geq t) \leq 2\exp\left(-c\min\left(\frac{t^2}{(\mathsf{E}\sup\limits_{A\in\mathscr{A}}\|AX\|)^2}, \frac{t}{\sup\limits_{A\in\mathscr{A}}\|A\|}\right)\right). \tag{4.2}$$

Apart from the new techniques the significance of this result is that previously (see, for example, Ledoux and Talagrand (2013)) similar bounds were one-sided and had a multiplicative constant greater than 1 before $\mathsf{E}Z$. These results are sometimes called *deviation inequlities* in contrast to the *concentration bounds* of the form (4.2) that will be studied below. A simplified proof of the upper-tail of (4.2) appeared later in Boucheron et al. (2003). Similar inequalities in the Gaussian case follow from the results in Borell (1984) and Arcones and Gine (1993).

Observe, that when the diagonal elements are zero, for each $A \in \mathscr{A}$ the corresponding quadratic form is centered, $\mathsf{E}\varepsilon^T A\varepsilon = 0$. In a general situation we will be interested in the analysis of

$$Z = \sup_{A\in\mathscr{A}} (X^\top AX - \mathsf{E}X^\top AX), \tag{4.3}$$

for a random vector X taking its values in $\mathbb{R}^n$. As before, the analysis of both the expectation and the concentration properties of this random variable appeared a lot in a recent literature. Just to name a few: Kramer et al. (2014) study $\mathsf{E}Z$ and deviations of Z for classes of positive semidefinite matrices with applications to compressive sensing, Dicker and Erdogdu (2017) prove deviation inequalities for $\sup_{A\in\mathscr{A}}(X^\top AX - \mathsf{E}X^\top AX)$ and subgaussian vectors X under some extra assumptions. Additionally, a recent paper Adamczak et al. (2018b) studies deviation bounds for $Z = \|X^\top AX - \mathsf{E}X^\top AX\|$ with Banach space-valued matrices A and Gaussian variables, providing upper and lower bounds for the moments. Finally, it was shown in Adamczak (2015) that if X satisfies the so-called *concentration property* with constant K, that is for every 1-Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and any $t \geq 0$ it holds $\mathsf{E}|\varphi(X)| < \infty$ and

$$\mathsf{P}\left(|\varphi(X) - \mathsf{E}\varphi(X)| \geq t\right) \leq 2\exp\left(-t^2/2K^2\right), \tag{4.4}$$

then the following bound (similar to (4.2)) holds for every $t \geq 0$

$$\mathsf{P}(|Z - \mathsf{E}Z| \geq t) \leq 2\exp\left(-c\min\left(\frac{t^2}{K^2(\mathsf{E}\sup\limits_{A\in\mathscr{A}}\|AX\|)^2}, \frac{t}{K^2\sup\limits_{A\in\mathscr{A}}\|A\|}\right)\right). \tag{4.5}$$

This result has an application in the covariance estimation and recovers another recent concentration result of Koltchinskii and Lounici (2017); we will discuss this in what follows. The drawback of (4.5) is that the concentration property is quite restrictive: it works when X has standard Gaussian distribution, for some log-concave distributions (see Ledoux (2001)), but at the same time does not hold for general subgaussian entries and even in the simplest case of Rademacher random vector ε.

We extend the mentioned results in two directions. On one hand we revisit the result of Boucheron et al. (2003) for bounded variables allowing non-zero diagonal values of the matrices, and on the other we allow unbounded subgaussian variables X_i. First, let us recall the following definition. For $\alpha > 0$ denote the ψ_α-norm of a random variable Y by

$$\|Y\|_{\psi_\alpha} = \inf\left\{t \geq 0 : \mathsf{E}\exp\left(\frac{|Y|^\alpha}{t^\alpha}\right) \leq 2\right\},$$

which is a proper norm whenever $\alpha \geq 1$. A random variable Y with $\|Y\|_{\psi_1} < \infty$ will be refereed to as subexponential and $\|Y\|_{\psi_2} < \infty$ will be refereed to as subgaussian and the corresponding norm is usually named a subgaussian norm. We also use the $L_p(P)$ norm. For $p \geq 1$ we set $\|Y\|_{L_p} = (\mathsf{E}|Y|^p)^{\frac{1}{p}}$. One of our main contributions is the following upper-tail bound.

Theorem 4.1. *Suppose that components of $X = (X_1, \ldots, X_n)$ are independent centered random variables and $\mathscr{A}$ is a finite family of $n \times n$ real symmetric matrices. Denote $M = \left\|\max_i |X_i|\right\|_{\psi_2}$. Then, for any $t \geq \max\{M\mathsf{E}\sup\|AX\|, M^2\sup_A\|A\|\}$ it holds*

$$\mathsf{P}(Z - \mathsf{E}Z \geq t) \leq \exp\left(-c\min\left(\frac{t^2}{M^2(\mathsf{E}\sup\limits_{A\in\mathscr{A}}\|AX\|)^2}, \frac{t}{M^2\sup\limits_{A\in\mathscr{A}}\|A\|}\right)\right),$$

where $c > 0$ is an absolute constant and Z is defined by (4.3).

Remark 4.1. *In Theorem 4.1 and below we assume that all $A \in \mathscr{A}$ is symmetric. This was done only for the convenience of presentation and in fact, the analysis may be performed for general square matrixes. The only difference will be that in many places A should be replaced by $\frac{1}{2}(A + A^T)$.*

In particular, Theorem 4.1 recovers the right-tail of the result of Talagrand (4.2) up to absolute constants, since in this case we obviously have $\left\|\max_i |\varepsilon_i|\right\|_{\psi_2} \lesssim 1$. Furthermore, the result of Theorem 4.1 works without the assumption used in Talagrand (1996) and

Boucheron et al. (2003) that diagonals of all matrices in $\mathscr{A}$ are zero. Moreover, it is also applicable in some situations when the concentration property (4.4) holds: indeed, if X is a standard normal vector in $\mathbb{R}^n$ then it is well known (see Ledoux and Talagrand (2013)) that $M = \left\|\max_i |X_i|\right\|_{\psi_2} \sim \sqrt{\log n}$ and at the same time if the identity matrix $I_n \in \mathscr{A}$ then $\mathsf{E}\sup_{A\in\mathscr{A}} \|AX\| \ge \mathsf{E}\|X\| \gtrsim \sqrt{n}$. Therefore, in this case the factor M is only of at most logarithmic order when compared to $\mathsf{E}\sup_{A\in\mathscr{A}} \|AX\|$.

In a special case when $\mathscr{A}$ consists of just one matrix our bound recovers the bound which is similar to the original Hanson-Wright inequality. On the one hand our bound may have an extra logarithmic factor that depends on the dimension n. On the other hand the original term $\max_i \|X_i\|_{\psi_2}\|A\|_{\mathrm{HS}}$ is replaced by the better term $\mathsf{E}\|AX\|$. We will discuss this phenomenon below. The core of the proof of the Hanson-Wright inequality in Rudelson and Vershynin (2013) is based on the decoupling technique which may be used (at least in a straightforward way) to prove the deviation, but not the concentration inequality for $\sup_{A\in\mathscr{A}}(X^\top AX - \mathsf{E}X^\top AX)$ in the case when $\mathscr{A}$ consists of more than one matrix.

A natural question to ask is whether one may improve Theorem 4.1 and replace $M = \left\|\max_i |X_i|\right\|_{\psi_2}$ by $K = \max_i\left\|X_i\right\|_{\psi_2}$. In what follows we discuss that in the deviation version of Theorem 4.1 this replacement is not possible in some cases. This is quite unexpected in light of the fact that $\left\|\max_i |X_i|\right\|_{\psi_2}$ does not appear in the original Hanson-Wright inequality. Therefore, we believe that the form of our result is close to optimal. We also provide the following extension of Theorem 4.1, which may be better in some cases.

Proposition 4.1. *Suppose that components of $X = (X_1,\dots,X_n)$ are independent centered random variables. Suppose also, that the variables X_i have symmetric distribution (X_i has the same distribution as $-X_i$). Let $\mathscr{A}$ be a finite family of $n\times n$ real symmetric matrices. Denote $M = \left\|\max_i |X_i|\right\|_{\psi_2}$ and $K = \max_i\left\|X_i\right\|_{\psi_2}$ and let $\mathbf{g}$ be a standard Gaussian vector in $\mathbb{R}^n$. Then, for any $t \ge \max\{MK\mathsf{E}\sup\|AG\|, MK\sup_A\|A\|\}$ it holds*

$$\mathsf{P}(Z - \mathsf{E}Z \ge t) \le \exp\left(-c\min\left(\frac{t^2}{M^2K^2(\mathsf{E}\sup\limits_{A\in\mathscr{A}}\|AG\|)^2}, \frac{t}{MK\sup\limits_{A\in\mathscr{A}}\|A\|}\right)\right),$$

where $c > 0$ are absolute constants and Z is defined by (4.3).

Remark 4.2. *Proposition 4.1 is closer to the standard Hanson-Wright inequality* (4.1). *Indeed, in the case when $\mathscr{A} = \{A\}$ we have $\mathsf{E}\|AG\| \sim \|A\|_{HS}$. The difference is that K^4 and K^2 are replaced by M^2K^2 and MK respectively.*

We proceed with some notations that will be used below. For a non-negative random variable Y, define its *entropy* as

$$\mathrm{Ent}(Y) = \mathsf{E}Y \log Y - \mathsf{E}Y \log \mathsf{E}Y.$$

Instead of the concentration property (4.4) we also discuss the following property:

Assumption 4.1. *We say that the random vector X taking its values in $\mathbb{R}^n$ satisfies the logarithmic Sobolev inequality with constant $K > 0$ if for any continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ it holds*

$$\mathrm{Ent}(f^2) \le 2K^2 \mathsf{E}\|\nabla f(X)\|^2, \tag{4.6}$$

whenever both sides of the inequality are not infinite.

To show that logarithmic Sobolev property is closely related to the concentration property we remind (Theorem 5.3 Ledoux (2001)) that Assumption 4.1 implies the concentration property (4.4) and the proof of this fact is based essentially on taking $f(X) = \exp(\lambda(\varphi(X) - \mathsf{E}\varphi(X))/2)$ for $\lambda > 0$ which implies

$$\mathrm{Ent}\left(\exp(\lambda(\varphi(X) - \mathsf{E}\varphi(X)))\right) \le \frac{K^2\lambda^2}{2} \mathsf{E}\exp(\lambda(\varphi(X) - \mathsf{E}\varphi(X))).$$

This is known to imply (4.4) through Herbst argument, see Boucheron et al. (2013). Moreover, the last inequality is equivalent to concentration property. Indeed, from the concentration property we know that $\|\varphi(X) - \mathsf{E}\varphi(X)\|_{\psi_2} \lesssim K$ and this implies (see van Handel (2016)) that for all $\lambda \in \mathbb{R}$

$$\mathrm{Ent}(\exp(\lambda(\varphi(X) - \mathsf{E}\varphi(X)))) \lesssim K^2\lambda^2 \mathsf{E}\exp(\lambda(\varphi(X) - \mathsf{E}\varphi(X))).$$

One of our technical contributions is that we use a similar scheme to prove Theorem 4.1 and to recover (4.5) under the logarithmic Sobolev Assumption 4.1. The application of logarithmic Sobolev inequalities requires computation of the gradient of the function of interest, that is in our case the gradient of $f(X) = \sup_{A \in \mathscr{A}}(X^T AX - \mathsf{E}X^T AX)$. It appears that in the analysis we need to control the behaviour of $\nabla f(X)$ (or its analogs) and, as in Boucheron et al. (2003) and Adamczak (2015), we will use a truncation argument to do so. However, in both cases our proofs will pass through the *entropy variational formula* of Boucheron et al. (2013), that states that for random variables Y, W with $\mathsf{E}\exp(W) < \infty$ it holds

$$\mathsf{E}(W\exp(\lambda Y)) \le \mathsf{E}\exp(\lambda Y)\log(\mathsf{E}\exp(W)) + \mathrm{Ent}(\exp(\lambda Y)). \tag{4.7}$$

This will allow us to shorten the proofs and avoid some technicalities appearing in previous papers. Finally, to prove Theorem 4.1 we use a second truncation argument: that will be based on Hoffman-Jørgensen inequality (see Ledoux and Talagrand (2013)). We also present two lemmas, which will be used several times in the text. Both results have short proofs and may be of independent interest.

Lemma 4.1. *Suppose, that for random variables Z, W and any $\lambda > 0$ it holds*

$$\mathrm{Ent}(e^{\lambda Z}) \le \lambda^2 \mathsf{E}We^{\lambda Z} \qquad \textit{and} \qquad \mathsf{P}(W > L + \theta t) \le e^{-t}, \tag{4.8}$$

where θ, L are positive constants. Then, the following concentration result holds

$$\mathsf{P}(Z - \mathsf{E}Z > t) \le \exp\left(-c\min\left\{\frac{t^2}{L+\theta}, \frac{t}{\sqrt{\theta}}\right\}\right), \tag{4.9}$$

where $c > 0$ is an absolute constant. Moreover, if (4.8) *holds as well for $\lambda \le 0$, we have*

$$\mathsf{P}(|Z - \mathsf{E}Z| > t) \le 2\exp\left(-c\min\left\{\frac{t^2}{L+\theta}, \frac{t}{\sqrt{\theta}}\right\}\right).$$

The second technical result is a version of the convex concentration inequality of Talagrand (1996), which does not require the boundedness of components of X.

Lemma 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex, L-Lipschitz function with respect to Euclidian norm in $\mathbb{R}^n$ and $X = (X_1, \ldots, X_n)$ be a random vector with independent components. Then, it holds for any $t \ge CL\left\|\max_i |X_i|\right\|_{\psi_2}$*

$$\mathsf{P}\left(|f(X) - \mathsf{E}f(X)| > t\right) \le \exp\left(-c\frac{t^2}{L^2\left\|\max_i |X_i|\right\|_{\psi_2}^2}\right),$$

where $c, C > 0$ are absolute constants.

We discuss the optimality of this result in what follows. Finally, we sum up the structure of the rest of this chapter and outline the main contributions:

- Section 4.1 is devoted to applications and discussions and consists of several parts. At first, we give a simple proof of the uniform bound of Adamczak (2015) under the logarithmic Sobolev assumption. The second paragraph is devoted to improvements in the non-uniform Hanson-Wright inequality (4.1) in the subgaussian regime. Furthermore, we apply our techniques to obtain a uniform concentration result similar to Theorem 4.1 in a particular case of non-independent components. We consider the Ising model under Dobrushin's condition that caught some attention recently (see Adamczak et al. (2018a) and Götze et al. (2018)). The question we study was raised by Marton (2003) in a closely related scenario. Finally, we show that it is not possible in general to replace $\|\max_i |X_i|\|_{\psi_2}$ with $\max_i \|X_i\|_{\psi_2}$ in Theorem 4.1 by providing an appropriate counterexample.

- In Section 4.2 we present the proof of Theorem 4.1. Between the lines, we prove Lemma 4.8 and Lemma 4.2. Finally, we give a proof of Proposition 4.1.
- In Section 4.3 we prove a dimension-free matrix Bernstein inequality that holds for random matrices with the subexponential spectral norm. The proof is based on the same truncation approach as in the proof of Theorem 4.1. We demonstrate how our Bernstein inequality can be used in the context of covariance estimation for subgaussian observations, improving the state-of-the-art result of Lounici (2014) for covariance estimation with missing observations.

4.1 Some applications and discussions

We begin with some notation that will be used below. For a random vector X taking its values in $\mathbb{R}^n$ let $X_1, \dots, X_n$ denote its components. In the case when all the components of X are independent let X_i' denote the independent copy of the component X_i. Symbol $\sim$ denotes equivalence up to absolute constants and $\lesssim$ denotes an inequality up to some absolute constant. The numbers $C, c > 0$ denote absolute constants, which also may change from line to line.

A uniform Hanson-Wright inequality under the logarithmic Sobolev condition

In this paragraph we recover the result of Adamczak (2015) under the Assumption 4.1. Consider a random variables Z defined by (4.3) as a function of X, that satisfies logarithmic Sobolev assumption (4.6).

Following Adamczak (2015) we assume without the loss of generality, that $\mathscr{A}$ is a finite set of matrices, then Z is Lebesgue-a.e. differentiable and

$$\|\nabla Z(X)\| \leq 2 \sup_A \|AX\|,$$

bounded by a Lipschitz function of X with good concentration properties.

Remark 4.3. *Note, that Assumption 4.1 applies only for smooth functions, so that a standard smoothing argument should be used (see e.g. Ledoux (2001)). For sake of completeness we recover this argument in Section 4.4. In what follows in this section we assume that none of these potential technical problems appear.*

In particular, since X satisfies log-Sobolev condition with constant K, we have (Theorem 5.3 in Ledoux (2001))

$$\mathsf{P}\left(\sup_A \|AX\| \geq \mathsf{E}\sup_A \|AX\| + K\sqrt{t}\sup_A \|A\|\right) \leq e^{-t}.$$

Taking square and using $(a+b)^2 \leq 2a^2 + 2b^2$, we get

$$\mathsf{P}\left(\sup_A \|AX\|^2 \geq 2\left(\mathsf{E}\sup_A \|AX\|\right)^2 + 2K^2 \sup_A \|A\|^2 t\right) \leq e^{-t}.$$

Furthermore, the logarithmic Sobolev condition implies for any $\lambda \in \mathbb{R}$

$$\operatorname{Ent}(e^{\lambda Z}) \leq 4K^2\lambda^2 \mathsf{E}\sup_A \|AX\|^2 e^{\lambda Z}.$$

Therefore, by Lemma 4.1 it holds for any $t \geq 1$,

$$\mathsf{P}\left(|Z - \mathsf{E}Z| > C\left(K\mathsf{E}\sup_A \|AX\|\sqrt{t} + K^2 \sup_A \|A\| t\right)\right) \leq 2e^{-t},$$

which coincides with (4.5) for K-concentrated vectors up to absolute constant factors.

Remark 4.4. *This result may be used directly to prove the concentration for $\|\hat{\Sigma} - \Sigma\|$, where $\hat{\Sigma}$ is the sample covariance defined as $\hat{\Sigma} = \frac{1}{N}\sum_{i=1}^N X_i X_i^\top$ and $X_1, \ldots, X_N$ are centered Gaussian vectors with the covariance matrix Σ (see Theorem 4.1 in Adamczak (2015)). We return to the covariance estimation problem in Section 4.3.*

Improving Hanson-Wright inequality in the subgaussian regime

Our analysis implies, in particular, an improved version of Hanson-Wright inequality (4.1) in some cases. We consider a centered random vector $X = (X_1, \dots, X_n)$ with independent subgaussian components and set $K = \max_i \|X_i\|_{\psi_2}$, $M = \|\max_i |X_i|\|_{\psi_2}$. In this case (4.1) implies that with probability at least $1 - 2e^{-t}$ it holds

$$X^\top AX - \mathsf{E}X^\top AX \lesssim K^2 \left(\|A\|_{\mathrm{HS}}\sqrt{t} + \|A\|t\right). \tag{4.10}$$

At the same time, Theorem 4.1 for a single matrix $\mathscr{A} = \{A\}$ implies with the same probability

$$X^\top AX - \mathsf{E}X^\top AX \lesssim M\mathsf{E}\|AX\|\sqrt{t} + M^2\|A\|t. \tag{4.11}$$

Observe that when $|X_i| \le L$ almost surely for each $i \le n$, we have $M \lesssim \min\{K\sqrt{\log n}, L\}$. The following example illustrates the difference between these two bounds.

Example 4.1. *Assume, $\delta = (\delta_1, \dots, \delta_n)$ is a sequence of independent Bernoulli random variables with the mean δ and let $\delta \le \frac{1}{4}$. For $X = (\delta_1 - \delta, \dots, \delta_n - \delta)$ we easily get*

$$\mathsf{E}\|AX\| \le \sqrt{\mathsf{E}X^T A^2 X} \le \sqrt{\delta}\|A\|_{HS}.$$

On the other hand, for $\delta \le \frac{1}{4}$ it holds

$$\begin{aligned}\|X_1\|_{\psi_2}^2 = \|\delta_1 - \delta\|_{\psi_2}^2 &\sim \sup_{\lambda \in \mathbb{R}} \frac{\log(\mathsf{E}\exp(\lambda(\delta_1 - \delta)))}{\lambda^2} \\ &= \sup_{\lambda \in \mathbb{R}} \frac{\log(\delta\exp(\lambda(1-\delta)) + (1-\delta)\exp(-\lambda\delta))}{\lambda^2} = \frac{1-2\delta}{4\log((1-\delta)/\delta)} \sim \frac{1}{|\log\delta|},\end{aligned}$$

where the last line follows directly from Theorem 1.1 in Schlemm (2016). Therefore, the standard Hanson-Wright inequality implies that with probability at least $1 - e^{-t}$ it holds,

$$X^\top AX - \mathsf{E}X^\top AX \lesssim \frac{1}{|\log\delta|}\left(\|A\|_{HS}\sqrt{t} + \|A\|t\right),$$

while (4.11) *and $M \lesssim \min\{K\sqrt{\log n}, 1\}$ imply that for $t \ge 1$ and $\delta \le \frac{1}{4}$ it holds with probability at least $1 - 2e^{-t}$*

$$X^\top AX - \mathsf{E}X^\top AX \lesssim \min\left\{\sqrt{\frac{\delta\log n}{|\log\delta|}}, \sqrt{\delta}\right\}\|A\|_{HS}\sqrt{t} + \min\left\{\frac{\log n}{|\log\delta|}, 1\right\}\|A\|t. \tag{4.12}$$

It is easy to verify that $\lim_{\delta \to 0+} \sqrt{\delta}|\log \delta| = 0$*, thus the inequality* (4.12) *is better than Hanson-Wright inequality for this* X *in the subgaussian regime (when the* t*-term is dominated by the* $\sqrt{t}$*-term).*

Uniform concentration results in the Ising model

Suppose, we have a discrete random vector $\sigma \in \{-1, 1\}^n$ with the distribution defined by

$$\pi(\sigma) = \frac{1}{Z'} \exp\left(\sum_{i,j=1}^{n} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{n} h_i\sigma_i \right),$$

where Z' is a normalizing factor. This distribution defines the *Ising model* with parameters $J = (J_{ij})_{i,j=1}^n$ and $\mathbf{h} = (h_i)_{i=1}^n$.

For an arbitrary function f on $\{-1, 1\}^n$ denote a difference operator,

$$|\mathfrak{d} f|^2(\sigma) = \frac{1}{2}\sum_{i=1}^{n} (f(\sigma) - f(T_i\sigma))^2 \pi(-\sigma_i \mid \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots),$$

where the operator $T_i\sigma = (\sigma_1, \dots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \dots)$ flips the sign of the ith coordinate, and $\pi(\cdot \mid \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots)$ is conditional distribution of the ith coordinate, given the rest of the elements. The following recent result provides log-Sobolev inequality for vector σ under Dobrushin-type conditions.

Theorem 4.2 (Proposition 1.1, Götze et al. (2018))**.** *Suppose,* $\|\mathbf{h}\|_\infty \le \alpha$ *and* J *satisfies* $J_{ii} = 0$ *and*

$$\|J\|_{1 \mapsto 1} = \max_{i=1,\dots,n} \sum_{j=1}^{n} |J_{ij}| \le 1 - \rho \tag{4.13}$$

There is a constant $C = C(\alpha, \rho)$*, such that for an arbitrary function* f *on* $\{-1, 1\}^n$ *it holds,*

$$\mathrm{Ent}(f^2) \le 2C\mathsf{E}|\mathfrak{d} f|^2.$$

Remark 4.5. *Following Götze et al. (2018) the condition* (4.13) *will be called* Dobrushin's condition.

We may obtain the following uniform concentration result which is a simple outcome of our Lemma 4.1 and Theorem 4.2.

Proposition 4.2. *Let $\mathscr{A}$ be a finite set of symmetric matrices with zero diagonal. It holds in the Ising model under Dobrushin's condition and $\|\mathbf{h}\|_\infty \le \alpha$ that for any $t \ge 0$*

$$\mathsf{P}\left(\sup_{A\in\mathscr{A}} \sigma^\top A\sigma - \mathsf{E}\sup_{A\in\mathscr{A}} \sigma^\top A\sigma \ge t\right) \le \exp\left(-c\min\left(\frac{t^2}{(\mathsf{E}\sup\limits_{A\in\mathscr{A}}\|A\sigma\| + \sup_{A\in\mathscr{A}}\|A\|)^2}, \frac{t}{\sup\limits_{A\in\mathscr{A}}\|A\|}\right)\right), \tag{4.14}$$

where C depends only on α, ρ.

Proof. Let $\sigma'_{(i)} = (\sigma_1, \dots, \sigma_{i-1}, \sigma'_i, \sigma_{i+1}, \dots)$ given all but the i-th element, the variables σ_i and σ'_i are independent and are distributed according to $\pi(\cdot \mid \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots)$. Obviously, we may have all $\sigma_1, \dots, \sigma_i$ and $\sigma'_1, \dots, \sigma'_n$ defined on the same discrete probability space, and thus we will use the notation $\pi(\cdot)$ and $\pi(\cdot \mid \cdot)$ for the distribution and the conditional distribution. Then, we have

$$\begin{aligned}
\mathsf{E}|\mathfrak{d}f|^2(\sigma) &= \frac{1}{2}\sum_{i=1}^n \mathsf{E}(f(\sigma) - f(T_i\sigma))^2 \pi(-\sigma_i \mid \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots) \\
&= \sum_{i=1}^n \sum_{\sigma\in\{-1,1\}^n} \pi(\sigma) \sum_{\sigma'_i\in\{-1,1\}} (f(\sigma) - f(\sigma'_{(i)}))_+^2 \pi(\sigma'_i \mid \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots)
\end{aligned}$$

where we switched from $\frac{1}{2}(f(\sigma) - f(\sigma'_{(i)}))^2$ to $(f(\sigma) - f(\sigma'_{(i)}))_+^2$ due to the symmetry between σ_i and σ'_i.

Observe, that denoting for short $\sigma^{-i} = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$ and using the independence of σ_i and σ'_i given σ^{-i}, we have $\pi(\sigma_i, \sigma'_i \mid \sigma^{-i}) = \pi(\sigma_i \mid \sigma^{-i})\pi(\sigma'_i \mid \sigma^{-i})$, and therefore by the chain rule,

$$\begin{aligned}
\pi(\sigma)\pi(\sigma'_i \mid \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots) &= \pi(\sigma^{-i})\pi(\sigma_i \mid \sigma^{-i})\pi(\sigma'_i \mid \sigma^{-i}) \\
&= \pi(\sigma^{-i})\pi(\sigma_i, \sigma'_i \mid \sigma^{-i}) = \pi(\sigma'_i, \sigma_i, \sigma^{-i}).
\end{aligned}$$

Finally, we get

$$\mathsf{E}|\mathfrak{d}f|^2(\sigma) = \sum_{i=1}^n \sum_{(\sigma,\sigma'_i)\in\{-1,1\}^{n+1}} (f(\sigma) - f(\sigma'_{(i)}))_+^2 \pi(\sigma, \sigma'_i) = \sum_{i=1}^n \mathsf{E}(f(\sigma) - f(\sigma'_{(i)}))_+^2.$$

Now we want to consider the function

$$Z = \sup_{A\in\mathscr{A}} \sigma^\top A\sigma, \tag{4.15}$$

where $\mathscr{A}$ is a given set of symmetric matrices with zero diagonal (the diagonal is not important here, since $\sigma_i^2 = 1$). Applying Theorem 4.2 to $f = e^{\lambda Z/2}$, we have

$$\begin{aligned}\mathsf{E}|\mathfrak{d} f|^2(\sigma) &= \mathsf{E}\sum_{i=1}^n (f(\sigma) - f(\sigma'_{(i)}))_+^2 = \mathsf{E}e^{\lambda Z}\sum_{i=1}^n (1 - e^{\lambda(Z(\sigma)-Z(\sigma'_{(i)}))/2})_+^2 \\ &\le \frac{\lambda^2}{4}\mathsf{E}e^{\lambda Z}\sum_{i=1}^n (Z - Z(\sigma'_{(i)}))_+^2,\end{aligned}$$

where for $\widetilde{A}$ being maximizer of (4.15) we have,

$$\begin{aligned}\sum_{i=1}^n (Z - Z(\sigma'_{(i)}))_+^2 &\le \sum_{i=1}^n \left(\sigma^\top \widetilde{A}\sigma - [\sigma'_{(i)}]^\top \widetilde{A}\sigma'_{(i)}\right)_+^2 = \sum_{i=1}^n \left(2(\sigma_i - \sigma'_i)\sum_{j=1}^n \widetilde{A}_{ij}\sigma_j\right)_+^2 \\ &\le 16 \sup_{A\in\mathscr{A}} \|A\sigma\|^2.\end{aligned}$$

Note, that concentration for $\sup_{A\in\mathscr{A}} \|A\sigma\|$ is implied by the same result. Indeed, we have

$$\begin{aligned}\sum_{i=1}^n \left(\sup_{A\in\mathscr{A},\gamma\in S^{n-1}} \gamma^\top A\sigma - \sup_{A\in\mathscr{A},\gamma\in S^{n-1}} \gamma^\top A\sigma'_{(i)}\right)_+^2 &\le \sum_{i=1}^n (\tilde{w}^\top\sigma - \tilde{w}^\top\sigma'_{(i)})_+^2 \\ &= \sum_{i=1}^n (\tilde{w}_i(\sigma_i - \sigma'_i))_+^2 \le 4\sup_A \|A\|,\end{aligned}$$

where $\widetilde{w}^\top = \gamma^\top A$ is such that $\sup_A \|A\sigma\| = \widetilde{w}^\top\sigma$. Thus, the expectation of corresponding difference operator is bounded by $4\sup_A \|A\|$, so that due to standard Herbst argument, Theorem 4.2 implies

$$\mathsf{P}\left(\sup_{A\in\mathscr{A}} \|A\sigma\| > \mathsf{E}\sup_{A\in\mathscr{A}} \|A\sigma\| + C\sup\|A\|\sqrt{t}\right) \le e^{-t}.$$

To sum up, by Theorem 4.2 it holds,

$$\mathrm{Ent}(e^{\lambda Z}) \le \lambda^2 \mathsf{E}(4\sup_{A\in\mathscr{A}} \|A\sigma\|)e^{\lambda Z}.$$

It is left to apply Lemma 4.1, which brings us to a uniform Hanson-Wright-type concentration bound for the Ising model

$$\mathsf{P}\left(\sup_A \sigma^\top A\sigma - \mathsf{E}\sup_A \sigma^\top A\sigma > C(\sqrt{t}\mathsf{E}\sup_A \|A\sigma\| + (\sqrt{t}+t)\sup_A \|A\|)\right) \ge 1 - e^{-t}, \quad (4.16)$$

where C only depends on α, ρ from Theorem 4.2. The claim follows. $\square$

Remark 4.6. *In the case when $\mathscr{A} = \{A\}$ our result implies the upper tail of the recent concentration inequality proved in Adamczak et al. (2018a) (see Theorem 2.2 and Example 2.5). To show this fact (denoting $\overline{\sigma} = \sigma - \mathsf{E}\sigma$) we observe that*

$$\mathsf{E}\|A\sigma\| \leq \mathsf{E}\|A\overline{\sigma}\| + \|A\mathsf{E}\sigma\| = \mathsf{E}\|A\overline{\sigma}\| + \big(\sum_{i=1}^{n}(\sum_{j=1}^{n} A_{i,j}\mathsf{E}\sigma_j)^2\big)^{\frac{1}{2}}.$$

Now, it is well known that $\mathrm{Ent}(f^2) \leq 2c\mathsf{E}|\mathfrak{d}f|^2$ *implies Poincaré's inequlity* $\mathrm{Var}(f) \leq c\mathsf{E}|\mathfrak{d}f|^2$ *and therefore,*

$$\|\mathsf{E}\overline{\sigma}\,\overline{\sigma}^{\top}\| = \sup_{u \in S^{n-1}} \mathrm{Var}(u^T\overline{\sigma}) \leq (c(\alpha,\rho)/2) \sup_{u \in S^{n-1}} 4\|u\|^2 = 2c(\alpha,\rho).$$

This implies,

$$\mathsf{E}\|A\overline{\sigma}\|^2 = \mathsf{E}\,\mathrm{tr}(A^2\overline{\sigma}\,\overline{\sigma}^{\top}) \leq \|A\|_{HS}^2 \|\mathsf{E}\overline{\sigma}\,\overline{\sigma}^T\| \leq 2c(\rho,\alpha)\|A\|_{HS}^2,$$

where we used that $\mathrm{tr}(BD) \leq \mathrm{tr}(B)\|D\|$, *which holds for any symmetric and nonnegative* B, D. *Finally,*

$$\|A\sigma\| \leq C(\rho,\alpha)\|A\|_{HS} + \big(\sum_{i=1}^{n}(\sum_{j=1}^{n} A_{i,j}\mathsf{E}\sigma_j)^2\big)^{\frac{1}{2}}.$$

The right-hand side term appears instead of $\|A\sigma\|$ *in Example 2.5 mentioned above.*

Replacing $\|\max_i |X_i|\|_{\psi_2}$ with $\max_i \|X_i\|_{\psi_2}$ in Theorem 4.1

Here we show that it is essentially not possible in general to substitute $\|\max_i |X_i|\|_{\psi_2}$ with $\max_i \|X_i\|_{\psi_2}$ in Theorem 4.1 by presenting a concrete counterexample, which was kindly suggested by Radosław Adamczak. Suppose the opposite, that there is an absolute constant $C > 0$ such that for any set of matrices $\mathscr{A}$ and any subgaussian random variables $X_1, \ldots, X_n$ it holds with probability at least $1 - e^{-t}$,

$$Z \leq C\left(\mathsf{E}Z + \max_i \|X_i\|_{\psi_2}\sqrt{t}\,\mathsf{E}\sup_A \|AX\| + \max_i \|X_i\|_{\psi_2}^2 \sup_A \|A\| t\right), \tag{4.17}$$

which implies with some other constant $C' > 0$

$$\mathsf{E}^{1/2} Z^2 \le C' \left(\mathsf{E} Z + \max_i \|X_i\|_{\psi_2} \mathsf{E} \sup_A \|AX\| + \max_i \|X_i\|_{\psi_2}^2 \sup_A \|A\| \right).$$

Notice, that here we also allow a constant in front of the expectation.

Let us take $\mathscr{A} = \{A^{(1)}, \ldots, A^{(n)}\}$ with $A^{(i)}$ having only one nonzero element $A_{ii}^{(i)} = 1$. For simplicity take i.i.d. $X_1, \ldots, X_n$ with $\mathsf{E} X_i^2 = 1$, so that

$$Z = \max_{i \le n} (X_i^2 - 1), \qquad \sup_A \|AX\| = \max_{i \le n} |X_i|, \qquad \sup_A \|A\| = 1.$$

Then, assuming, say $\|X_1\|_{\psi_2} \le 4$ we have

$$\left\| \max_i X_i^2 - 1 \right\|_{L_2} \le C' \left(\mathsf{E} \max_i (X_i^2 - 1) + 4 \mathsf{E} \max_i |X_i| + 16 \right),$$

which since $\| \max_i X_i^2 \|_{L_1} \ge \|X_i\|_{L_2} = 1$ implies

$$\left\| \max_i X_i^2 \right\|_{L_2} \le 1 + C' (\| \max_i X_i^2 \|_{L_1} + 4 \mathsf{E} \max_i |X_i| + 15) \le (1 + 20C') \| \max_i X_i^2 \|_{L_1}.$$

Note, that this inequality also holds if we rescale $X_i' = \alpha X_i$ for an arbitrary $\alpha > 0$. Therefore, if we have a moment equivalence $\|X_1\|_{\psi_2} \le 4 \|X_1\|_{L_2}$, we can always rescale to have $\|X_1\|_{L_2} = 1$ and $\|X_1\|_{\psi_2} \le 4$, so that the above inequality holds.

Taking the latter into account, we conclude that there is a constant $D > 0$, such that if a centred random X_1 satisfies $\|X_1\|_{\psi_2} \le 4 \|X_1\|_{L_2}$, then for any $n \ge 1$ the following holds,

$$\left\| \max_{i \le n} X_i^2 \right\|_{L_2} \le D \| \max_{i \le n} X_i^2 \|_{L_1}. \tag{4.18}$$

It is known that such hypercontractivity of maxima implies certain regularity of tails of the distribution of X_1^2. In this case by Theorem 4.6 in Hitczenko et al. (1998) for any $\rho, \varepsilon > 0$ there is another constant $A = A(D, \rho, \varepsilon) > 1$ such that for all $t \ge t_0 = \rho \|X_1\|_{L_1}$ it holds,

$$A^q \mathsf{P}(X_1^2 > At) \le \varepsilon \mathsf{P}(X_1^2 > t),$$

so that in our case of $p = 2$ and $q = 1$ and taking $\rho = \varepsilon = 1$, there is $A = A(D) > 1$ such that for all $t \ge \|X_1\|_{L_1}$ it holds

$$\mathsf{P}(X_1^2 > At) \le \frac{1}{A} \mathsf{P}(X_1^2 > t). \tag{4.19}$$

The latter does not have to hold for any subgaussian random variable X_1. For instance, taking a symmetric random variable X_1 with $\mathsf{P}(|X_1| = 1) = 1 - e^{-r}$ and $\mathsf{P}(|X_1| = \sqrt{r}) = e^{-r}$ for $r \geq 4 > 4\log 2$ we have $\mathsf{E}\exp\left(\frac{|X_1|^2}{2}\right) = e^{\frac{1}{2}}(1 - e^{-r}) + e^{-r+\frac{r}{2}} \leq e^{\frac{1}{2}} + e^{-\frac{r}{2}} \leq 2$, which implies $\|X_1\|_{\psi_2} \leq 2$. Moreover, for $r \geq 4$ we also have $\mathsf{E}X_1^2 \geq 1 - e^{-\frac{r}{2}} \geq \frac{1}{2}$, thus $\|X_1\|_{L_2} \geq 1/\sqrt{2}$ and the conditions of (4.18) are satisfied. But for large enough $r > At$ for $t = t_0$, we have

$$\mathsf{P}\left(X_1^2 > At\right) = \mathsf{P}(X_1^2 > t) = e^{-r},$$

therefore breaking the tail regularity (4.19). Thus, it is impossible to establish inequality of form (4.17). We also note that it is also possible to prove that (4.18) may not hold for X_1 defined above via some direct computations.

By the same reason it is not possible to replace $\|\max_{i\leq n}|X_i|\|_{\psi_2}$ with $\max_{i\leq n}\|X_i\|_{\psi_2}$ in Lemma 4.2. Indeed, suppose for any convex L-Lipschitz function f it holds,

$$\mathsf{P}\left(f(X) \leq C(\mathsf{E}f(X) + L\max_{i\leq n}\|X_i\|_{\psi_2}\sqrt{t})\right) \leq e^{-t}.$$

Then, taking $f(X) = \max_{i\leq n}|X_i|$, which is convex and 1-Lipschitz, we get

$$\left\|\max_{i\leq n} X_i^2\right\|_{L_2} = \left\|\max_{i\leq n}|X_i|\right\|_{L_4} \leq C'\left(\mathsf{E}\max_i |X_i| + \max_i \|X_i\|_{\psi_2}\right),$$

which for the same random variable X_1 as before implies (4.18) and leads to a contradiction.

4.2 Proof of Theorem 4.1

In this section we assume that all components of X are independent. We recall that X_i' denotes an independent copy of the component X_i. The main tool of the proof is the modified logarithmic Sobolev inequality (see Theorem 2 in Boucheron et al. (2003) or Theorem 6.15 in Boucheron et al. (2013)). Set,

$$Z_i' = Z(X_1, \ldots, X_{i-1}, X_i', X_i, \ldots, X_n).$$

Then, by symmetrised version of the inequality we have for any λ,

$$\mathrm{Ent}(e^{\lambda Z}) \leq \sum_{i=1}^{n} \mathsf{E}e^{\lambda Z}\tau(-\lambda(Z - Z_i')_+),$$

where $\tau(x) = x(e^x - 1)$. Since $\tau(x) \leq x^2$ for $x \leq 0$, we have for all $\lambda \geq 0$,

$$\mathrm{Ent}(e^{\lambda Z}) \leq \lambda^2 \mathsf{E} V_+ e^{\lambda Z}, \qquad V_+ := \mathsf{E}' \sum_{i=1}^n (Z - Z_i')_+^2.$$

The right-hand side of the inequality can be "decoupled" by variational entropy formula (4.7), as it is done in the proof of Lemma 4.1, that we presented in the introduction.

Proof of Lemma 4.1. We have

$$\mathrm{Ent}(e^{\lambda Z}) \leq \lambda^2 L e^{\lambda Z} + \lambda^2 \mathsf{E}(W - L)_+ e^{\lambda Z}.$$

Due to the deviation bound for W it holds for some absolute constant $C > 0$,

$$\mathsf{E} \exp\left(\frac{(W - L)_+}{C\theta}\right) \leq e.$$

Therefore, by (4.7) we have,

$$\mathsf{E}(W - L)_+ / (C\theta) e^{\lambda Z} \leq \mathsf{E} e^{\lambda Z} + \mathrm{Ent}(e^{\lambda Z}),$$

which implies

$$(1 - C\theta\lambda^2)\mathrm{Ent}(e^{\lambda Z}) \leq \lambda^2 (L + C\theta) \mathsf{E} e^{\lambda Z}.$$

By the Herbst argument (see e.g., Proposition 6.1 in Boucheron et al. (2013)) we have for each $0 < \lambda \leq (C\theta)^{-1/2}$,

$$\log \mathsf{E} \exp(\lambda(Z - \mathsf{E}Z)) \leq 2(L + C\theta)\lambda^2,$$

therefore $(Z - \mathsf{E}Z)$ is subexponential and the right-hand concentration bound follows. If (4.8) holds for all $\lambda \in \mathbb{R}$, the two sided inequality can be derived in the same way.

□

Remark 4.7. *Note, there is as well a moment version of the modified log-Sobolev inequality, see Theorem 2 in Boucheron et al. (2005b). By the theorem it holds, for all $q \geq 2$*

$$\|(Z - \mathsf{E}Z)_+\|_{L_q} \leq \sqrt{2\kappa q} \|\sqrt{V_+}\|_{L_q},$$

where $\kappa < 2$ is an absolute constant. Then, if we have a condition for V_+ in the form

$$\|\sqrt{V_+}\|_{L_q} \leq \sqrt{L} + \sqrt{\theta q}, \qquad \forall q \geq 2, \tag{4.20}$$

which is equivalent to the second inequality in (4.8) *up to constants, then it simply holds for each $q \geq 2$*

$$\|(Z - \mathsf{E}Z)_+\|_{L_q} \leq \sqrt{4Lq} + \sqrt{4\theta}q,$$

which as well implies (4.9) *up to constants. We note that similar moment computations where used in Boucheron et al. (2005b) to analyze the Rademacher chaos. Similarly, one can introduce the moment analog of logarithmic Sobolev inequality (see equation 3 in Adamczak and Wolff (2015)):*

$$\|Z(X) - \mathsf{E}Z(X)\|_{L_q} \leq K\sqrt{q}\||\nabla Z(X)|\|_{L_q},$$

where $K > 0$ is a constant, $|\cdot|$ stands for the standard Euclidian norm and $q \geq 2$. Now, if it holds (which may be in some cases derived by the second application of the moment analog of logarithmic Sobolev inequalities)

$$\||\nabla Z(X)|\|_{L_q} \leq \mathsf{E}|\nabla Z(X)| + \||\nabla Z(X)| - \mathsf{E}|\nabla Z(X)|\|_{L_q} \leq \sqrt{L} + K\sqrt{\theta q}, \qquad \forall q \geq 2$$

then

$$\|Z - \mathsf{E}Z\|_{L_q} \leq K(\sqrt{Lq} + K\sqrt{\theta}q),$$

which implies the bound similar to (4.5).

Now we establish a version of our result that does not require neither centered X_i nor that they have variance one. In this case it might happen that $\mathsf{E}X^\top AX \neq \mathrm{tr}(A)$, but in fact the value we subtract does not really affect the concentration properties. In general we can consider,

$$Z = \sup_{A \in \mathscr{A}} (X^\top AX - g(A)), \tag{4.21}$$

where $g : \mathbb{R}^{n \times n} \to \mathbb{R}$ is an arbitrary function.

Lemma 4.3. *Suppose, $|X_i| \leq K$ almost surely, are independent, but not necessary centred. Then, for Z defined by* (4.21) *and for any $t \geq 1$*

$$Z - \mathsf{E}Z \leq C\left(K(\mathsf{E}\sup_A \|AX\| + \mathsf{E}\sup_A \|\operatorname{Diag}(A)X\|)\sqrt{t} + K^2 \sup_A \|A\|t\right),$$

with probability at least $1 - e^{-t}$ where C is an absolute constant.

Proof. Let $\widetilde{A}$ be the matrix on which the maximum is achieved for the original sample. We have,

$$\begin{aligned}\sum_{i\le n}(Z-Z_i)_+^2 &\le \sum_{i\le n}\left(2(X_i-X_i')\sum_{j\ne i}\tilde{a}_{ij}X_j+\tilde{a}_{ii}(X_i^2-X_i'^2)\right)^2\\ &=\sum_{i\le n}(X_i-X_i')^2\left(2\sum_{j\ne i}\tilde{a}_{ij}X_j+\tilde{a}_{ii}(X_i+X_i')\right)^2\\ &\le (2K)^2\sum_{i\le n}\left(2\sum_{j}\tilde{a}_{ij}X_j+\tilde{a}_{ii}(X_i'+X_i)\right)^2,\end{aligned}$$

where the last line follows from $|X_i-X_i'|\le 2K$. Applying the triangle inequality we get

$$V_+=\mathsf{E}'\sum_{i\le n}(Z-Z_i)_+^2\le (2K)^2\mathsf{E}'\sup_A(2\|AX\|+\|\operatorname{Diag}(A)X\|+\|\operatorname{Diag}(A)X'\|)^2,$$

where the expectation is taken with respect to the copy sample. Thus,

$$\mathsf{E}'V_+\le 12K^2\left(\sup_A\|AX\|^2+\sup_A\|\operatorname{Diag}(A)X\|^2+\mathsf{E}\sup_A\|\operatorname{Diag}(A)X\|^2\right).$$

Since $|X_i|\le K$, we have by convex concentration for Lipshitz functions (see e.g. Theorem 6.10 in Boucheron et al. (2013))

$$\mathsf{P}\left(\sup_A\|AX\|>\mathsf{E}\sup_A\|AX\|+2\sqrt{2}K\sup_A\|A\|\sqrt{t}\right)\le e^{-t}. \tag{4.22}$$

Using $(a+b)^2\le 2a^2+2b^2$ we have, that for $L\sim(K\mathsf{E}\sup\|AX\|+K\mathsf{E}\sup\|\operatorname{Diag}(A)X\|)^2$ and $\theta\sim(K\sup\|A\|)^2$ it holds

$$\mathsf{P}\left(V_+>L+\theta t\right)\le e^{-t},$$

so that due to the modified log-Sobolev inequality (4.2) we can use Lemma 4.1. This provides us with the inequality

$$\mathsf{P}(Z-\mathsf{E}Z>C(\sqrt{L+\theta}\sqrt{t}+\sqrt{\theta t}))\le e^{-t},$$

where we can neglect the θ in front of $\sqrt{t}$ when $t\ge 1$. □

Note, that here we have the term $\mathsf{E}\sup_A \|\operatorname{Diag}(A)X\|$, which can be avoided in the case of centered variables X_i, therefore matching the previous bounds (4.5) and (4.2).

Corollary 4.1. *Suppose, $|X_i| \le K$ almost surely and $\mathsf{E}X_i = 0$. Then, for any $t > 0$*

$$Z - \mathsf{E}Z \lesssim K\mathsf{E}\sup_A \|AX\|\sqrt{t} + K^2 \sup_A \|A\|t,$$

with probability at least $1 - 2e^{-t}$.

In the next two lemmas we show how to get rid of the diagonal term, which finishes the proof of the corollary above.

Lemma 4.4. *Suppose, $Y \in \mathbb{R}^n$ has i.i.d. coordinates with symmetric distribution, and let $\mathscr{B}$ be a set of $n \times n$ positive-definite symmetric matrices. Then,*

$$\mathsf{E}\sup_{B\in\mathscr{B}} Y^\top \operatorname{Diag}(B)Y \le \mathsf{E}\sup_{B\in\mathscr{B}} Y^\top BY.$$

Proof. Given the vector $x \in \mathbb{R}^n$ let $\operatorname{Diag}(x)$ denote a diagonal $n \times n$ matrix with x on diagonal. Since $Y \overset{d}{=} \operatorname{Diag}(\varepsilon)Y$ for an independent Rademacher vector $\varepsilon \in \{1,-1\}^n$, we have by Jensen's inequality

$$\begin{aligned} \mathsf{E}\sup_{B\in\mathscr{B}} Y^\top BY &= \mathsf{E}\mathsf{E}_\varepsilon \sup_{B\in\mathscr{B}} Y^\top \operatorname{diag}(\varepsilon)B\operatorname{diag}(\varepsilon)Y \\ &\ge \mathsf{E}\sup_{B\in\mathscr{B}} \mathsf{E}_\varepsilon Y^\top \operatorname{diag}(\varepsilon)B\operatorname{diag}(\varepsilon)Y \\ &= \mathsf{E}\sup_{B\in\mathscr{B}} Y^\top \operatorname{Diag}(B)Y, \end{aligned}$$

where E_ε denotes expectation conditioned on Y. □

Lemma 4.5. *For a random X with independent mean zero coordinates it holds,*

$$\mathsf{E}\sup_{A\in\mathscr{A}} \|\operatorname{Diag}(A)X\| \le C\mathsf{E}\sup_{A\in\mathscr{A}} \|AX\|,$$

where $C > 0$ is an absolute constant.

Proof. Setting X' as an independent copy of X, we have a standard symmetrisation argument, i.e. applying first Jensen's and then the triangle inequality we have,

$$\mathsf{E}\sup_{A\in\mathscr{A}} \|AX\| \le \mathsf{E}\sup_{A\in\mathscr{A}} \|A(X-X')\| \le 2\mathsf{E}\sup_{A\in\mathscr{A}} \|AX\|. \tag{4.23}$$

Observe that $X - X' \overset{d}{=} (X - X')\operatorname{diag}(\varepsilon) = \operatorname{diag}(X - X')\varepsilon$, where $\varepsilon \in \{1, -1\}^n$ is an independent Rademacher vector. Therefore, we have

$$\mathsf{E} \sup_{A \in \mathscr{A}} \|A(X - X')\| = \mathsf{E}\mathsf{E}_\varepsilon \sup_{A \in \mathscr{A}} \|A \operatorname{diag}(X - X')\varepsilon\|,$$

where E_ε denotes the expectation with respect to ε. Conditionally on $(X - X')$ set $\mathscr{A}_{X,X'} = \{A \operatorname{diag}(X - X') : A \in \mathscr{A}\}$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be columns of a matrix A. Notice, that for any matrix A we have $\operatorname{Diag}(A^\top A) = \operatorname{diag}(\|\mathbf{a}_1\|^2, \ldots, \|\mathbf{a}_n\|^2) \succeq \operatorname{diag}(A_{11}^2, \ldots, A_{nn}^2) = \operatorname{Diag}(A)^2$. Therefore, by Lemma 4.4

$$\mathsf{E}_\varepsilon \sup_{A \in \mathscr{A}_{X,X'}} \|\operatorname{Diag}(A)\varepsilon\|^2 \le \mathsf{E}_\varepsilon \sup_{A \in \mathscr{A}_{X,X'}} \|A\varepsilon\|^2. \tag{4.24}$$

We now want to get rid of the squares in the inequality above which is possible due to concentration. Let us fix some matrix $B \in \mathscr{B}$, where $\mathscr{B}$ is a set of matrixes. Then, $\mathsf{E}\|B\varepsilon\|^2 = \|B\|_{HS}^2$ and by Khinchin's inequality it holds

$$\mathsf{E}\|B\varepsilon\| \ge \frac{1}{\sqrt{2}} \|B\|_{HS},$$

with the optimal constant due to Szarek (1976). Thus, we have

$$\mathsf{E} \sup_{B \in \mathscr{B}} \|B\varepsilon\| \ge \sup_{B \in \mathscr{B}} \mathsf{E}\|B\varepsilon\| \ge \frac{1}{\sqrt{2}} \sup_{B \in \mathscr{B}} \|B\|.$$

Note furthermore, that by the convex Poincare inequality (Theorem 3.17, Boucheron et al. (2013)) it holds,

$$\operatorname{Var}(\sup_{B \in \mathscr{B}} \|B\varepsilon\|) = \mathsf{E} \sup_{B \in \mathscr{B}} \|B\varepsilon\|^2 - \left(\mathsf{E} \sup_{B \in \mathscr{B}} \|B\varepsilon\|\right)^2 \le 4 \sup_{B \in \mathscr{B}} \|B\|^2.$$

Therefore, it holds $\mathsf{E}\sup_B \|B\varepsilon\|^2 \le (\mathsf{E}\sup_{B \in \mathscr{B}} \|B\varepsilon\|)^2 + 4\sup_B \|B\|^2 \le 9\,(\mathsf{E}\sup_{B \in \mathscr{B}} \|B\varepsilon\|)^2$ and we get

$$(\mathsf{E} \sup_{B \in \mathscr{B}} \|B\varepsilon\|)^2 \le \mathsf{E} \sup_{B \in \mathscr{B}} \|B\varepsilon\|^2 \le 9(\mathsf{E} \sup_{B \in \mathscr{B}} \|B\varepsilon\|)^2.$$

The last inequality combined with (4.24) implies

$$\mathsf{E}_\varepsilon \sup_{A \in \mathscr{A}_{X,X'}} \|\operatorname{Diag}(A)\varepsilon\| \le \left(\mathsf{E}_\varepsilon \sup_{A \in \mathscr{A}_{X,X'}} \|\operatorname{Diag}(A)\varepsilon\|^2\right)^{\frac{1}{2}} \le 3\mathsf{E}_\varepsilon \sup_{A \in \mathscr{A}_{X,X'}} \|A\varepsilon\|.$$

Now, taking an expectation with respect to X, X' and applying (4.23) again we finish the proof. $\square$

4.2.1 Truncation for unbounded variables

In this section we finish the proof of Theorem 4.1. In order to apply the bounded version, we want to truncate each variable X_i, which can be done by the approach from Adamczak (2008) (see reference therein for more details on the applications of this method), where it was used in the context of Talagrand's concentration inequality. Suppose, $\|\max_i |X_i|\|_{\psi_2} < \infty$ and set

$$Y_i = X_i \mathbf{I}(|X_i| \le M), \qquad W_i = X_i - Y_i, \tag{4.25}$$

with $M = 8\mathsf{E}\max|X_i|$. We have,

$$\begin{aligned} Z &= \sup_A (Y^\top AY - \mathsf{E}X^\top AX + W^\top AX + W^\top AY) \\ &\le \sup_A (Y^\top AY - \mathsf{E}X^\top AX) + \sup_A |W^\top AX| + \sup_A |W^\top AY| \\ &\le \sup_A (Y^\top AY - \mathsf{E}X^\top AX) + \|W\| \sup_A \|AX\| + \|W\| \sup_A \|AY\|. \end{aligned} \tag{4.26}$$

Now that the variables Y_i are bounded by the value M pointwise, the first term of the last line can be carried out by Lemma 4.3.

To bound the rest we need to control the deviations of $\|W\|$. We have, $\|W\|^2 = W_1^2 + \cdots + W_n^2$ is a sum of independent variables with bounded ψ_1-norm, so we can control it's expectation via Hoffman-Jørgensen inequality. Due to the choice of the cut-off, we have by Markov inequality,

$$\mathsf{P}\left(\max_i W_i^2 > 0\right) = \mathsf{P}\left(\max_i |X_i| > M\right) \le \frac{\mathsf{E}\max_i |X_i|}{M} \le \frac{1}{8}.$$

Denoting $S_k = W_1^2 + \cdots + W_k^2$ we have $\|W\|^2 = S_n$. Then,

$$P(\max_{k \le n} |S_k| > 0) \le P(\max_{i \le n} W_i^2 > 0) \le \frac{1}{8}.$$

Therefore, by Proposition 6.8 in Ledoux and Talagrand (2013) it holds,

$$\mathsf{E}\|W\|^2 = \mathsf{E}S_n \le 8\mathsf{E}\max_{i \le n} W_i^2 \lesssim \|\max_{i \le n} |X_i|\|_{\psi_2}^2,$$

where the latter holds since $\|\max_{i \le n} W_i^2\|_{\psi_1} \le \|\max_i |X_i|\|_{\psi_2}^2$. Furthermore, by Theorem 6.21 in Ledoux and Talagrand (2013) it holds with some absolute constant K_1,

$$\begin{aligned}\left\|\sum_{i=1}^{n} W_i^2 - \mathsf{E}W_i^2\right\|_{\psi_1} &\le K_1 \left(\mathsf{E}\big|\|W\|^2 - \mathsf{E}\|W\|^2\big| + \big\|\max_i |W_i^2 - \mathsf{E}W_i^2|\big\|_{\psi_1}\right) \\ &\le 2K_1 \left(\mathsf{E}\|W\|^2 + \big\|\max_i W_i^2\big\|_{\psi_1}\right) \\ &\lesssim \|\max_i |X_i|\|_{\psi_2}^2,\end{aligned}$$

and given the bound on the expectation of $\|W\|^2$ it implies,

$$\big\|\|W\|\big\|_{\psi_2} \lesssim \big\|\max_i |X_i|\big\|_{\psi_2}.$$

Hence we obtain the deviation bound for any $t > 0$,

$$\mathsf{P}\left(\|W\| \ge C\sqrt{t}\|\max_i |X_i|\|_{\psi_2}\right) \le 2e^{-t}. \tag{4.27}$$

Now we apply Lemma 4.3 to the bounded variables Y. Notice, that the theorem does not require the variables to be centered, we only use it in the Corollary 4.1. Taking this into account, the lemma applies to the variables Y in the following form. Set $g(A) = \mathsf{E}X^\top AX$ and $Z(Y) = \sup_A (Y^\top AY - g(A))$, then by Lemma 4.3 it holds,

$$Z(Y) - \mathsf{E}Z(Y) \lesssim M\sqrt{t}(\mathsf{E}\sup_A \|AY\| + \mathsf{E}\sup_A \|\operatorname{Diag}(A)Y\|) + M^2 t \sup_A \|A\| \tag{4.28}$$

with probability at least $1 - e^{-t}$. Next all we need to do is to carefully replace the expectations $\mathsf{E}Z(Y)$, $\mathsf{E}\sup_A \|AY\|$ and $\mathsf{E}\sup_A \|\operatorname{Diag}(A)Y\|$ in (4.28) by those, taken with respect to X, as in the original formulation of the result.

First we want to provide a concentration bound for the convex function $\sup_A \|AX\|$, that accounts for unbounded variables. As a matter of fact we prove the following Lemma which is even slightly stronger than Lemma 4.2.

Lemma 4.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be separately convex*[1] *L-Lipschitz with respect to Euclidian norm in $\mathbb{R}^n$ and $X = (X_1, \ldots, X_n)$ be a random vector with independent components. Then, it*

[1] This means that for every $i = 1, \ldots, n$ it is a convex function of i-th variable if the rest of the variables are fixed. Any convex function is separately convex.

holds for $t \geq 1$

$$\mathsf{P}\left(f(X) > \mathsf{E}f(X) + C\big\|\max_i |X_i|\big\|_{\psi_2} L\sqrt{t}\right) \leq e^{-t},$$

where $C > 0$ *is an absolute constant. Additionally, if* f *is convex L-Lipschitz, then for any* $t > 0$

$$\mathsf{P}\left(|f(X) - \mathsf{E}f(X)| > C\big\|\max_i |X_i|\big\|_{\psi_2} L\sqrt{t}\right) \leq 4e^{-t}.$$

Proof. By the convex concentration (Theorem 6.10 in Boucheron et al. (2013)) for bounded Y_i defined by (4.25) it holds for any $t > 0$

$$\mathsf{P}\left(f(Y) > \mathsf{E}f(Y) + C\|\max_i |X_i|\|_{\psi_2} L\sqrt{t}\right) \leq e^{-t}.$$

Moreover, due to the Lipschitz assumption and (4.27) we have

$$|f(X) - f(Y)| \leq L\|W\| \lesssim L\|\max_i |X_i|\|_{\psi_2}\sqrt{1+t},$$

where the latter holds with probability at least $1 - e^{-t}$. Integrating these two bounds we also get

$$|\mathsf{E}f(X) - \mathsf{E}f(Y)| \lesssim L\|\max_i |X_i|\|_{\psi_2}, \tag{4.29}$$

which together implies that with probability at least $1 - e^{-t}$ it holds

$$\begin{aligned} f(X) - \mathsf{E}f(X) &\leq f(Y) - \mathsf{E}f(Y) + |f(X) - f(Y)| + |\mathsf{E}f(X) - \mathsf{E}f(Y)| \\ &\lesssim L\|\max_i |X_i|\|_{\psi_2}\sqrt{t}. \end{aligned}$$

The proof of the lower tail bound follows from Theorem 7.12 in Boucheron et al. (2013) and the standard relation between median and the expectation, which holds in our case. □

From the lemma it follows due to the fact that $\sup_A \|AX\|$ if $\sup_A \|A\|$-Lipschitz we have

$$\mathsf{P}\left(\sup_A \|AX\| > \mathsf{E}\sup_A \|AX\| + C\big\|\max_i |X_i|\big\|_{\psi_2} \sup_A \|A\|\sqrt{t}\right) \leq 2e^{-t}. \tag{4.30}$$

Moreover, similar to (4.29) it holds

$$\left|\mathsf{E}\sup_A \|AY\| - \mathsf{E}\sup_A \|AX\|\right| \lesssim C\|\max_i |X_i|\|_{\psi_2} \sup_A \|A\|. \tag{4.31}$$

Next, we bound the difference between $\mathsf{E}Z(X)$ and $\mathsf{E}Z(Y)$.

Lemma 4.7. *It holds*

$$|\mathsf{E}Z(Y) - \mathsf{E}Z(X)| \lesssim \|\max_i |X_i|\|_{\psi_2} \mathsf{E}\sup_A \|AX\| + \|\max_i |X_i|\|_{\psi_2}^2 \sup_A \|A\|.$$

Proof. Similarly to (4.26),

$$\begin{aligned}|\mathsf{E}Z(Y) - \mathsf{E}Z(X)| &\le \mathsf{E}\|W\| \sup_A \|AX\| + \mathsf{E}\|W\| \sup_A \|AY\| \\ &\le \mathsf{E}^{1/2}\|W\|^2 (\mathsf{E}^{1/2} \sup_A \|AX\|^2 + \mathsf{E}^{1/2} \sup_A \|AY\|^2),\end{aligned} \tag{4.32}$$

where by (4.27) $\mathsf{E}^{1/2}\|W\|^2 \lesssim \|\max_i |X_i|\|_{\psi_2}$ and by (4.30),

$$\mathsf{E}\sup_A \|AX\|^2 \lesssim \left(\mathsf{E}\sup_A \|AX\|\right)^2 + \|\max_i |X_i|\|_{\psi_2}^2 \sup_A \|A\|^2,$$

which taking square root turns into,

$$\mathsf{E}^{1/2} \sup_A \|AX\|^2 \lesssim \mathsf{E}\sup_A \|AX\| + \|\max_i |X_i|\|_{\psi_2} \sup_A \|A\|.$$

Similarly and using (4.31) we have,

$$\begin{aligned}\mathsf{E}^{1/2} \sup_A \|AY\|^2 &\lesssim \mathsf{E}\sup_A \|AY\| + \|\max_i |X_i|\|_{\psi_2} \sup_A \|A\| \\ &\lesssim \mathsf{E}\sup_A \|AX\| + \|\max_i |X_i|\|_{\psi_2} \sup_A \|A\|.\end{aligned}$$

Plugging it in (4.32) we get the required inequality. □

Therefore, in (4.28) we can replace by the lemma above

$$\mathsf{E}Z(Y) \le \mathsf{E}Z(X) + C\left(\|\max_i |X_i|\|_{\psi_2} \mathsf{E}\sup_A \|AX\| + \|\max_i |X_i|\|_{\psi_2}^2 \sup_A \|A\|\right), \tag{4.33}$$

and by Lemma 4.31 (neglecting the diagonal term for centred X due to Lemma 4.5)

$$\mathsf{E}\sup_A \|AY\| + \mathsf{E}\sup_A \|\operatorname{Diag}(A)Y\| \le C\left(\mathsf{E}\sup_A \|AX\| + \|\max_i |X_i|\|_{\psi_2} \sup_A \|A\|\right). \tag{4.34}$$

Finally, with probability at least $1-e^{-t}$ for $t \geq 1$ we have from (4.26), (4.31) and (4.30)

$$
\begin{aligned}
|Z(X)-Z(Y)| &\leq \|W\| \sup_A \|AY\| + \|W\| \sup_A \|AX\| \\
&\lesssim \|W\| \mathsf{E} \sup_A \|AX\| + \|W\| \| \max_i |X_i| \|_{\psi_2} \sup_A \|A\| \sqrt{t},
\end{aligned}
$$

which using (4.27) turns into

$$
|Z(X)-Z(Y)| \lesssim \| \max_i |X_i| \|_{\psi_2} \mathsf{E} \sup_A \|AX\| \sqrt{t} + \| \max_i |X_i| \|_{\psi_2}^2 \sup_A \|A\| t.
$$

Putting this together with (4.33) and (4.34) we finish the proof of Theorem 4.1.

4.2.2 Proof of Proposition 4.1

The proof is essentially based on the application of the next standard deviation bound instead of the concentration bound of (4.30) in the proof of Theorem 4.1. Since we did not find an exact reference we derive it here.

Lemma 4.8. *Suppose, $X_1, \ldots, X_n$ are independent centered random variables and $\mathscr{A}$ is a finite set of symmetric matrices. Let $\mathbf{g}$ be a standard normal vector in $\mathbb{R}^n$. Then, it holds with probability at least $1 - Ce^{-t}$ that*

$$
\sup_{A \in \mathscr{A}} \|AX\| \lesssim \max_i \left\| X_i \right\|_{\psi_2} \left(\mathsf{E} \sup_{A \in \mathscr{A}} \|A\mathbf{g}\| + \sup_A \|A\| \sqrt{t} \right),
$$

where $C > 0$ is an absolute constant.

Proof. At first we observe that $\sup_{A \in \mathscr{A}} \|AX\| = \sup\limits_{A \in \mathscr{A}, \gamma \in S^{n-1}} \gamma^T AX$. Consider the metric ρ defined by $\rho(a,b) = \|a-b\| \max\limits_i \|X_i\|_{\psi_2}$ for any $a, b \in \mathbb{R}^n$. By Theorem 2.2.26 in Talagrand (2014b) it holds for $t \geq 0$ and an absolute constant $C > 0$ that with probability at least $1 - C\exp(-t)$

$$
\sup_{A \in \mathscr{A}, \gamma \in S^{n-1}} \gamma^T AX \lesssim \operatorname{diam}(\mathscr{A} S^{n-1}, \rho) \sqrt{t} + \gamma_2(\mathscr{A} S^{n-1}, \rho),
$$

where $\operatorname{diam}(\mathscr{A} S^{n-1}) = \sup\limits_{x,y \in \mathscr{A} S^{n-1}} \|x-y\| \max\limits_i \|X_i\|_{\psi_2} \leq 2 \sup\limits_{A \in \mathscr{A}} \|A\| \max_i \left\| X_i \right\|_{\psi_2}$ and the functional γ_2 is also defined in Talagrand (2014b). For the sake of brevity, we will not introduce its definition here. Finally, applying Talagrand's majorizing measure theorem (Theorem 2.4.1 in Talagrand (2014b)) we have

$$\gamma_2(\mathscr{A}S^{n-1},\rho) \lesssim \max_i \left\|X_i\right\|_{\psi_2} \mathsf{E} \sup_{x\in\mathscr{A}S^{n-1}} x^T G = \max_i \left\|X_i\right\|_{\psi_2} \mathsf{E} \sup_{A\in\mathscr{A}} \|AG\|.$$

The claim follows. □

Setting $M = 8\mathsf{E}\max_i|X_i|$ and $K = \max_i \|X_i\|_{\psi_2}$ consider the truncation scheme just like in (4.25). Due to the assumption that X_i have symmetric distribution, we have $\mathsf{E}Y_i = 0$, therefore the lemma above applies in the following form,

$$\mathsf{P}\left(\sup_{A\in\mathscr{A}} \|AY\| > CK(\mathsf{E}\sup_{A\in\mathscr{A}} \|A\mathbf{g}\| + \sup_A \|A\|\sqrt{t})\right) \le e^{-t},$$

which can be used instead of the convex concentration inequality (4.22) when dealing with modified log-Sobolev inequality, see proof of Lemma 4.3. Following this proof and using the fact that $\max_i |Y_i| \le M$ almost surely, we end up with the following concentration bound

$$Z(Y) - \mathsf{E}Z(Y) \lesssim MK\left(\mathsf{E}\sup_{A\in\mathscr{A}} \|A\mathbf{g}\|\sqrt{t} + \sup_A \|A\|t\right)$$

with probability at least $1-e^{-t}$ for any $t>1$. Furthermore, we can slightly modify the derivations of the previous section, again, using Lemma 4.8 instead of (4.30). In particular, we get with probability at least $1-e^{-t}$ for any $t>1$,

$$|Z(X) - Z(Y)| \lesssim MK(\mathsf{E}\sup_A \|AG\|\sqrt{t} + \sup_A \|A\|t),$$

and taking expectation we also get $|\mathsf{E}Z(X) - \mathsf{E}Z(Y)| \lesssim MK\mathsf{E}\sup_A \|AG\|$. The claim then follows from (4.26).

4.3 Matrix Bernstein inequality in the subexponential case

As we mentioned above, one of the prominent applications of the uniform Hanson-Wright inequalities is the recent concentration result in the Gaussian covariance estimation problem. It is known that covariance estimation problems may be alternatively approached by the matrix Bernstein inequality. Following the truncation approach, which was taken above we provide a version of matrix Bernstein inequality, that does not require uniformly bounded matrices. The standard version of the inequality (see Tropp (2012) and reference therein) may be formulated as follows: consider random independent matrices $X_1, \dots, X_N \in \mathbb{R}^{n\times n}$, such that almost surely $\max_i \|X_i\| \le L$. It holds

$$\mathsf{P}\left(\left\|\sum_{i=1}^N X_i - \mathsf{E}X_i\right\| > u\right) \le n\exp\left(-c\left(\frac{u^2}{\sigma^2}\bigwedge\frac{u}{L}\right)\right),$$

where c is an absolute constant and $\sigma^2 = \left\|\mathsf{E}\sum_{i=1}^N (X_i - \mathsf{E}X_i)^2\right\|$. The first problem with this result is that it does not hold in general cases when $\max_i \|X_i\|_{\psi_1}$ or $\max_i \|X_i\|_{\psi_2}$ are bounded. The second problem is the dependence on the dimension n, which does not allow applying it to operators in Hilbert spaces. For a positive definite real square matrix A we define the *effective rank* as $\tilde{\mathbf{r}}(A) = \frac{\mathrm{tr}(A)}{\|A\|}$. We show the following bound.

Proposition 4.3. *Suppose, we have random independent symmetric matrices $X_1, \dots, X_N \in \mathbb{R}^{n\times n}$, each satisfying $\left\|\|X_i\|\right\|_{\psi_1} < \infty$. Set $M = \left\|\max_{i\le N}\|X_i\|\right\|_{\psi_1}$ and let positive-definite matrix R be such that $\mathsf{E}\sum_{i=1}^N X_i^2 \preceq R$. Finally, set $\sigma^2 = \|R\|$. There are absolute constants $c, C, c_1 > 0$ such that for any $u \ge c_1 \max\{M, \sigma\}$ it holds*

$$\mathsf{P}\left(\left\|\sum_{i=1}^N X_i - \mathsf{E}X_i\right\| > u\right) \le C\tilde{\mathbf{r}}(R)\exp\left(-c\left(\frac{u^2}{\sigma^2}\bigwedge\frac{u}{M}\right)\right).$$

Remark 4.8. *Using the well known bound for the maximum of subexponential random variables (see Ledoux and Talagrand (2013)) we have*

$$\left\|\max_{i\le N}\|X_i\|\right\|_{\psi_1} \lesssim \log N \max_{i\le N}\left\|\|X_i\|\right\|_{\psi_1},$$

and so, up to constant factors. we may state the same bound for $M = \log N \max_{i\le N}\left\|\|X_i\|\right\|_{\psi_1}$. When $n = 1$ the effective rank plays no role and our bound recovers the version of classical Bernstein inequality which is due to Adamczak (2008). In this paper, it is also shown that the $\log N$ factor cannot be removed in general, meaning that $M = \left\|\max_{i\le N}\|X_i\|\right\|_{\psi_1}$ cannot be replaced by $\max_{i\le N}\left\|\|X_i\|\right\|_{\psi_1}$ in general.

Proof. Fix $U > 0$ and consider the decomposition

$$X_i = Y_i + Z_i, \qquad Y_i = X_i\mathbf{I}(\|X_i\| \le U), \qquad Z_i = X_i\mathbf{I}(\|X_i\| > U),$$

so that the matrices Y_i are uniformly bounded by U in operator norm. By the triangle inequality and the union bound,

$$\mathsf{P}\left(\left\|\sum_{i=1}^N X_i - \mathsf{E}X_i\right\| > 2u\right) \leq \mathsf{P}\left(\left\|\sum_{i=1}^N Y_i - \mathsf{E}Y_i\right\| > u\right) + \mathsf{P}\left(\left\|\sum_{i=1}^N Z_i - \mathsf{E}Z_i\right\| > u\right),$$

so the two parts can be treated separately. Throughout the proof $c > 0$ is an absolute constant which may change from line to line. It is known that uniformly bounded random matrices satisfy Bernstein-type inequality (see Theorem 3.1 in Minsker (2017)) for $u \geq \frac{1}{6}(U + \sqrt{U^2 + 36\sigma^2})$

$$\mathsf{P}\left(\left\|\sum_{i=1}^N Y_i - \mathsf{E}Y_i\right\| > u\right) \leq 14\tilde{\mathbf{r}}\left(\mathsf{E}\sum_{i=1}^N (Y_i - \mathsf{E}Y_i)^2\right)\exp\left(-\frac{cu^2}{\left\|\sum_{i=1}^N (Y_i - \mathsf{E}Y_i)^2\right\| + Uu}\right),$$

where we used $\|Y_i\| \leq U$. However, since we want to present this bound in terms of X_i and not Y_i, we need the following modification of the proof of Minsker's theorem. Using the notation of his proof, it follows from Lemma 3.1 in Minsker (2017):

$$\log \mathsf{E}\exp(\theta(Y_i - \mathsf{E}Y_i)) \preceq \frac{\phi(\theta U)}{U^2}\mathsf{E}(Y_i - \mathsf{E}Y_i)^2 \preceq \frac{\phi(\theta U)}{U^2}2\mathsf{E}Y_i^2 \preceq \frac{\phi(\theta U)}{U^2}2\mathsf{E}X_i^2.$$

Now, using the same lines of the proof, instead of formula (3.4) we have

$$\mathsf{E}\operatorname{tr}\phi\left(\theta\sum_{i=1}^N (Y_i - \mathsf{E}Y_i)\right) \leq \operatorname{tr}\left(\exp\left(\frac{\phi(\theta U)}{U^2}2\sum_{i=1}^N \mathsf{E}X_i^2\right) - I_d\right),$$

and lines (3.5) with the condition $\sum_{i=1}^n \mathsf{E}X_i^2 \preceq R$ imply

$$\exp\left(\frac{\phi(\theta U)}{U^2}2\sum_{i=1}^N \mathsf{E}X_i^2\right) - I_d \preceq \exp\left(\frac{2\phi(\theta U)}{U^2}R\right) - I_d \preceq \frac{R}{\sigma^2}\exp\left(\frac{2\phi(\theta U)}{U^2}\sigma^2\right),$$

where $\sigma^2 = \|R\|$. Following last lines of the proof of Theorem 3.1 we finally have

$$\mathsf{P}\left(\left\|\sum_{i=1}^N Y_i - \mathsf{E}Y_i\right\| > u\right) \leq 14\tilde{\mathbf{r}}(R)\exp\left(-\frac{cu^2}{\sigma^2 + Uu}\right), \tag{4.35}$$

for $u \geq C\max\{U, \sigma\}$.

We proceed with the analysis of Z_i. Set $U = 8\mathsf{E}\max_{i\le n}\|X_i\|$, then we have by Markov inequality

$$\mathsf{P}\left(\max_{k\le n}\left\|\sum_{i=1}^{k} Z_i\right\| > 0\right) \le \mathsf{P}\left(\max_{i\le n}\|Z_i\| > 0\right) = \mathsf{P}\left(\max_{i}\|X_i\| > U\right) \le 1/8.$$

Thus, we can apply Proposition 6.8 from Ledoux and Talagrand (2013) to Z_i taking values the Banach space $(\mathbb{R}^{n\times n}, \|\cdot\|)$ equipped with the spectral norm. We have,

$$\mathsf{E}\left\|\sum_{i=1}^{N} Z_i\right\| \le 8\mathsf{E}\max_{i\le n}\|Z_i\|,$$

which implies with some constant $K > 0$,

$$\mathsf{E}\left\|\sum_{i=1}^{N} Z_i - \mathsf{E}Z_i\right\| \le 2\mathsf{E}\left\|\sum_{i=1}^{N} Z_i\right\| \le 16\mathsf{E}\max_{i\le N}\|Z_i\| \le K\left\|\max_{i\le N}\|Z_i\|\right\|_{\psi_1}.$$

Using Theorem 6.21 from Ledoux and Talagrand (2013) in $(\mathbb{R}^{n\times n}, \|\cdot\|)$ we have,

$$\begin{aligned}\left\|\left\|\sum_{i=1}^{N} Z_i - \mathsf{E}Z_i\right\|\right\|_{\psi_1} &\le K_1\left(\mathsf{E}\left\|\sum_{i=1}^{N} Z_i - \mathsf{E}Z_i\right\| + \left\|\max_{i\le N}\|Z_i\|\right\|_{\psi_1}\right)\\ &\le K_2\left\|\max_{i\le N}\|Z_i\|\right\|_{\psi_1},\end{aligned}$$

with some constants $K_1, K_2 > 0$. This implies a deviation bound for $u \ge \left\|\max_{i\le N}\|Z_i\|\right\|_{\psi_1}$,

$$\mathsf{P}\left(\left\|\sum_{i=1}^{N} Z_i - \mathsf{E}Z_i\right\| > u\right) \le \exp\left(-\frac{cu}{\left\|\max_{i\le N}\|Z_i\|\right\|_{\psi_1}}\right),$$

where $c > 0$ is an absolute constant. Combining it with (4.35), and that for some absolute $C > 0$ we have $U \le C\left\|\max_{i\le N}\|X_i\|\right\|_{\psi_1}$ and $\left\|\max_{i\le N}\|Z_i\|\right\|_{\psi_1} \le \left\|\max_{i\le N}\|X_i\|\right\|_{\psi_1}$, we prove the claim. $\square$

To the best of our knowledge, the Proposition 4.3 is the first to combine two important properties: it simultaneously captures the effective rank instead of the dimension n and is valid for matrices with subexponential operator norm (previously matrix Bernstein inequality in the unbounded case was granted under the so-called Bernstein moment condition; we refer to Tropp (2012) and the references therein). We should also compare our results with

Proposition 2 of Koltchinskii (2011), which has the same form as our bound, but instead of the effective rank, the original dimension n is used and $M = \left\| \max_{i \le n} \|X_i\| \right\|_{\psi_1}$ is replaced by $\max_{i \le N} \left\| \|X_i\| \right\|_{\psi_1} \log \left(N \left(\max_{i \le N} \left\| \|X_i\| \right\|_{\psi_1} \right)^2 / \sigma^2 \right)$.

Application to covariance estimation with missing observations

Now we turn to the problem studied in Koltchinskii and Lounici (2017) and Lounici (2014). Suppose, we want to estimate the covariance structure of a centered random subgaussian vector $X \in \mathbb{R}^n$ (which will be assumed centered) based on N i.i.d. observations $X_1, \dots, X_N$. For the sake of brevity, we work with the finite-dimensional case, while as in Koltchinskii and Lounici (2017) our results will not depend explicitly on the dimension n. Recall, that a centered random vector $X \in \mathbb{R}^n$ is *subgaussian* if for all $u \in \mathbb{R}^n$ it holds

$$\|\langle X, u \rangle\|_{\psi_2} \lesssim (\mathsf{E}\langle X, u \rangle^2)^{\frac{1}{2}}, \tag{4.36}$$

which does not require any independence of components of X.

In what follows we discuss a more general framework suggested by Lounici (2014). Let $\delta_{i,j}$, $i \le N, j \le n$ be independent Bernoulli random variables with the mean δ. We assume that instead of observing $X_1, \dots, X_N$ we observe vectors $Y_1, \dots, Y_N$, which are defined as $Y_i^j = \delta_{i,j} X_i^j$. This means that some components of vectors $X_1, \dots, X_N$ are missing (replaced by zero) each with probability $1 - \delta$. Since δ can be easily estimated we assume that it is known. Following Lounici (2014), denote

$$\hat{\Sigma}^{(\delta)} = \frac{1}{N} \sum_{i=1}^{N} Y_i Y_i^{\top}.$$

It can be easily shown that the estimator

$$\hat{\Sigma} = (\delta^{-1} - \delta^{-2}) \operatorname{Diag}(\hat{\Sigma}^{(\delta)}) + \delta^{-2} \hat{\Sigma}^{(\delta)}$$

is an unbiased estimator of $\Sigma = \mathsf{E} X_i X_i^{\top}$. In particular,

$$\Sigma = (\delta^{-1} - \delta^{-2}) \operatorname{Diag}(\mathsf{E} Y_i Y_i^{\top}) + \delta^{-2} \mathsf{E} Y_i Y_i^{\top}. \tag{4.37}$$

Theorem 4.3. *Under the assumptions defined above, it holds with probability at least* $1-e^{-t}$ *for* $t \geq 1$

$$\|\hat{\Sigma}-\Sigma\| \lesssim \|\Sigma\| \max\left(\sqrt{\frac{\tilde{\mathbf{r}}(\Sigma)\log\tilde{\mathbf{r}}(\Sigma)}{N\delta^2}}, \sqrt{\frac{t}{N\delta^2}}, \frac{\tilde{\mathbf{r}}(\Sigma)(\log\tilde{\mathbf{r}}(\Sigma)+t)\log N}{N\delta^2}\right).$$

Remark 4.9. *The upper-bound above provides an important improvement upon Proposition 3 in Lounici (2014), which is*

$$\|\hat{\Sigma}-\Sigma\| \lesssim \|\Sigma\| \max\left(\sqrt{\frac{\tilde{\mathbf{r}}(\Sigma)\log n}{N\delta^2}}, \sqrt{\frac{\tilde{\mathbf{r}}(\Sigma)t}{N\delta^2}}, \frac{\tilde{\mathbf{r}}(\Sigma)(\log n+t)(\log N+t)}{N\delta^2}\right) \tag{4.38}$$

The bound (4.38) *depends on n and therefore is not applicable in the infinite dimensional scenarios. It also contains a term proportional to* t^2*, which appears due to a straightforward truncation of each observation. Moreover, this result has an unnecessary factor* $\tilde{\mathbf{r}}(\Sigma)$ *in the term* $\sqrt{\frac{\tilde{\mathbf{r}}(\Sigma)t}{N\delta^2}}$*. Finally, when* $\delta = 1$ *tighter results may be obtained using high probability generic chaining bounds for quadratic processes. In particular, Theorem 9 in Koltchinskii and Lounici (2017) implies*

$$\|\hat{\Sigma}-\Sigma\| \lesssim \|\Sigma\| \max\left(\sqrt{\frac{\tilde{\mathbf{r}}(\Sigma)}{N}}, \sqrt{\frac{t}{N}}, \frac{\tilde{\mathbf{r}}(\Sigma)}{N}, \frac{t}{N}\right) \tag{4.39}$$

Unfortunately, this analysis may not be implied for $\delta < 1$ *in general, since the assumption* (4.36) *will not hold for the vector Y, defined by* $Y_i^j = \delta_{i,j}X_i^j$*. Therefore, our technique is a reasonable alternative which works for general* δ *and is almost as tight as* (4.39) *when* $\delta = 1$.

To prove Theorem 4.3 we need the following technical Lemma, parts of which may as well be found in Lounici (2014). For a matrix A let $\mathrm{Diag}(A)$ denote its diagonal part and define $\mathrm{Off}(A) = A - \mathrm{Diag}(A)$.

Lemma 4.9. *Let* $X \in \mathbb{R}^n$ *satisfy* (4.36) *with covariance matrix* Σ *any* $Y = (\delta_1 X^1, \ldots, \delta_n X^n)$*, where* δ_i*,* $i \leq n$ *are independent Bernoulli random variables with the mean* δ*. Then, it holds*

$$\big\|\|\mathrm{Diag}(YY^\top)\|\big\|_{\psi_1} \lesssim \tilde{\mathbf{r}}(\Sigma)\|\Sigma\|, \qquad \big\|\|\mathrm{Off}(YY^\top)\|\big\|_{\psi_1} \lesssim \tilde{\mathbf{r}}(\Sigma)\|\Sigma\|.$$

Additionally, it holds for some absolute constant $C > 0$

$$\mathsf{E}\operatorname{Off}(YY^\top)^2 \preceq C\delta^2 \operatorname{tr}(\Sigma)(\Sigma + \operatorname{Diag}(\Sigma)), \quad \textit{and} \quad \mathsf{E}\operatorname{Diag}(YY^\top)^2 \lesssim C\delta \operatorname{tr}(\Sigma)\operatorname{Diag}(\Sigma). \tag{4.40}$$

Proof. Observe, that $\|\operatorname{Diag}(YY^\top)\| \le \|Y\|^2$ and $\|\operatorname{Off}(YY^\top)\| \le \|YY^\top\| + \|\operatorname{Diag}(YY^\top)\| \le 2\|Y\|^2$. Therefore,

$$\Big\| \|\operatorname{Off}(YY^\top)\| \Big\|_{\psi_1} \le 2\| \|Y\| \|_{\psi_2}^2 \le 2\| \|X\| \|_{\psi_2}^2 \lesssim \operatorname{tr}(\Sigma),$$

and the same bound holds for $\big\| \|\operatorname{Diag}(YY^\top)\| \big\|_{\psi_1}$.

Let A be an arbitrary symmetric matrix and let us calculate $\mathsf{E}(A \odot \delta\delta^\top)^2$, where $\odot$ denotes Hadamard product and $\delta = (\delta_1, \dots, \delta_n)$ is a vector with independent components having Bernoulli distribution with the mean δ. We have,

$$\Big[\mathsf{E}(A \odot \delta\delta^\top)^2\Big]_{ii} = \mathsf{E}\sum_k A_{ik}\delta_i\delta_k A_{ki}\delta_i\delta_k = \sum_k A_{ik}A_{ik}\mathsf{E}\delta_i^2\delta_k^2 = \delta^2[A^2]_{ii} + (\delta - \delta^2)A_{ii}^2.$$

For the element at the position ij with $i \ne j$ we have,

$$\begin{aligned}\Big[\mathsf{E}(A \odot \delta\delta^\top)^2\Big]_{ij} &= \mathsf{E}\sum_k A_{ik}\delta_i\delta_k A_{kj}\delta_j\delta_k = \sum_k A_{ik}A_{kj}\mathsf{E}\delta_i\delta_j\delta_k^2 \\ &= \delta^3[A^2]_{ij} + (\delta^2 - \delta^3)(A_{ii}A_{ij} + A_{ij}A_{jj}).\end{aligned}$$

This can be put together in the following expression,

$$\begin{aligned}\mathsf{E}(\delta\delta^\top \odot A)^2 = \delta^3 A^2 &+ (\delta^2 - \delta^3)\left[\operatorname{Diag}(A^2) + \operatorname{Off}(A)\operatorname{Diag}(A) + \operatorname{Diag}(A)\operatorname{Off}(A)\right] \\ &+ (\delta - \delta^2)\operatorname{Diag}(A)^2.\end{aligned}$$

Note, that all of these matrices are positive definite, apart from the term $\operatorname{Off}(A)\operatorname{Diag}(A) + \operatorname{Diag}(A)\operatorname{Off}(A)$, which we can obviously bound by $\frac{1}{2}(\operatorname{Off}(A) + \operatorname{Diag}(A))^2 = A^2/2$. Taking into account $\delta \le 1$, we have a simple bound

$$\begin{aligned}\mathsf{E}(\delta\delta^\top \odot A)^2 &\preceq \frac{1}{2}(\delta^3 + \delta^2)A^2 + (\delta^2 - \delta^3)\operatorname{Diag}(A^2) + (\delta - \delta^2)\operatorname{Diag}(A)^2 \\ &\preceq \delta^2(A^2 + \operatorname{Diag}(A^2)) + \delta\operatorname{Diag}(A)^2.\end{aligned}$$

Now recall that $Y = \mathrm{diag}(\delta)X$, therefore $\mathrm{Off}(YY^\top) = \delta\delta^\top \odot \mathrm{Off}(XX^\top)$. Since the latter has zero diagonal, the term with δ in the formula above disappears. Therefore,

$$\mathsf{E}\,\mathrm{Off}(YY^\top)^2 \preceq \delta^2 \left[\mathsf{E}\,\mathrm{Off}(XX^\top)^2 + \mathrm{Diag}\left(\mathsf{E}\,\mathrm{Off}(XX^\top)^2\right)\right]. \tag{4.41}$$

It holds $\mathsf{E}\,\mathrm{Off}(XX^T)^2 \preceq 2\mathsf{E}(XX^\top)^2 + 2\mathsf{E}\,\mathrm{Diag}(XX^T)^2$, and we also have from Lounici (2014) that $\mathsf{E}(XX^\top)^2 \preceq C\,\mathrm{tr}(\Sigma)\Sigma$. Additionally, due to subgaussianity (4.36) we have $\mathsf{E}X_i^4 \lesssim \Sigma_{ii}^2$. Finally, the following bound holds

$$\mathsf{E}\,\mathrm{Diag}(XX^\top)^2 \preceq C\,\mathrm{Diag}(\Sigma)^2 \preceq C\,\mathrm{tr}(\Sigma)\,\mathrm{Diag}(\Sigma).$$

Plugging this bounds into (4.41) we get the second inequality.

As for the diagonal, we have for $A = \mathrm{Diag}(XX^\top)$,

$$\mathsf{E}\,\mathrm{Diag}(YY^\top) \preceq 3\delta\mathsf{E}\,\mathrm{Diag}(XX^\top)^2 \preceq C\delta\,\mathrm{tr}(\Sigma)\,\mathrm{Diag}(\Sigma).$$

□

Lemma 4.10. *For Y as in Lemma 4.9 and any unit $u \in \mathbb{R}^n$ it holds,*

$$\|u^\top \mathrm{Off}(YY^\top)u\|_{L_2} \lesssim \delta^2\|\Sigma\|, \qquad \|u^\top \mathrm{Diag}(YY^\top)u\|_{L_2} \lesssim \delta\|\Sigma\|.$$

Proof. Let $v \in \mathbb{R}^n$ be as well arbitrary unit vector. First we want to check, that

$$\|u^\top \mathrm{Diag}(XX^\top)v\|_{L_4} \lesssim \|\Sigma\|, \qquad \|u^\top \mathrm{Off}(XX^\top)v\|_{L_4} \lesssim \|\Sigma\|. \tag{4.42}$$

Obviously, $\|u^\top XX^\top v\|_{L_4} \le \|u^\top X\|_{L_8}\|v^\top X\|_{L_8} \lesssim \|\Sigma\|$, so it is enough to check just for the diagonal. Let us apply simmetrization argument. Suppose, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)^\top$ are independent Rademacher variables, then

$$u^\top \mathrm{Diag}(XX^\top)v = \mathsf{E}_\varepsilon \varepsilon^\top \mathrm{diag}(u)XX^\top \mathrm{diag}(v)\varepsilon = \mathsf{E}_\varepsilon u_\varepsilon XX^\top v_\varepsilon,$$

where $u_\varepsilon = (u_1\varepsilon_1, \dots, u_d\varepsilon_d)^\top$ and E_ε denotes expectation conditioned on X. Then, by Jensen and Hölder inequalities,

$$\mathsf{E}\left(u^\top \mathrm{Diag}(XX^\top)u\right)^4 \le \mathsf{E}\left(u_\varepsilon^\top XX^\top u_\varepsilon\right)^4 = \mathsf{E}_\varepsilon \mathsf{E}^{1/2}[(u_\varepsilon^\top X)^8 \mid \varepsilon]\mathsf{E}^{1/2}[(v_\varepsilon^\top X)^8 \mid \varepsilon] \lesssim \|\Sigma\|^4,$$

thus implying (4.42).

Next, let us consider a zero diagonal symmetric matrix B. We have,

$$\mathsf{E}(\delta^\top B\delta)^2 = \sum_{i\neq j} B_{ij} \sum_{k\neq l} B_{kl} \mathsf{E}\delta_i\delta_j\delta_k\delta_l$$

Given $i \neq j$ and $k \neq l$ we have,

$$\begin{aligned}\mathsf{E}\delta_i\delta_j\delta_k\delta_l &= \delta^4 + (\delta^3 - \delta^4)\{\mathbf{I}(i=l) + \mathbf{I}(j=l) + \mathbf{I}(i=k) + \mathbf{I}(j=k)\} \\ &\quad + (\delta^2 - 2\delta^3 + \delta^4)\{\mathbf{I}((i,j)=(k,l)) + \mathbf{I}((i,j)=(l,k))\}.\end{aligned}$$

Therefore, due to the fact that B is symmetric we have

$$\mathsf{E}(\delta^\top B\delta)^2 = \delta^4 \left(\sum_{ij} B_{ij}\right)^2 + 4(\delta^3 - \delta^4)\sum_{ijk} B_{ij}B_{jk} + 2(\delta^2 - 2\delta^3 + \delta^4)\sum_{ij} B_{ij}^2$$

Denoting $\mathscr{S}(A) = \sum_{ij} A_{ij}$, we have $\left(\sum_{ij} B_{ij}\right)^2 = \mathscr{S}(B)^2$ and $\sum_{ijk} B_{ij}B_{jk} = \mathscr{S}(B^2)$. Thus,

$$\mathsf{E}(\delta^\top B\delta)^2 \lesssim \delta^4\mathscr{S}(B)^2 + \delta^3\mathscr{S}(B^2) + \delta^2\|B\|_{HS}^2$$

Since $u^\top \operatorname{Off}(YY^\top)u = \delta^\top \operatorname{diag}(u)\operatorname{Off}(XX^\top)\operatorname{diag}(u)\delta$ we have for $B = \operatorname{diag}(u)\operatorname{Off}(XX^\top)\operatorname{diag}(u)$,

$$\begin{aligned}\mathsf{E}\mathscr{S}(B)^2 &= \mathsf{E}\left(\mathbf{u}^\top \operatorname{Off}(YY^\top)v\right)^2 \leq \|\mathbf{u}^\top \operatorname{Off}(YY^\top)v\|_{L_4} \lesssim \|\Sigma\|^2, \\ \mathsf{E}S(B^2) &= \sum_i u_i^2 \mathsf{E}\left(u^\top \operatorname{Off}(XX^\top)\mathbf{e}_i\right)^2 \lesssim \|\Sigma\|^2,\end{aligned}$$

Finally,

$$\begin{aligned}\mathsf{E}\|B\|_{HS}^2 = \operatorname{tr}(B^2) &= \sum_i \mathbf{e}_i^\top \operatorname{diag}(u)\operatorname{Off}(XX^\top)\operatorname{diag}(u)^2\operatorname{Off}(XX^\top)\operatorname{diag}(u)\mathbf{e}_i \\ &= \sum_i u_i^2\mathbf{e}_i^\top \operatorname{diag}(u)\operatorname{Off}(XX^\top)\left[\sum_j u_j^2\mathbf{e}_j\mathbf{e}_j^\top\right]\operatorname{Off}(XX^\top)\operatorname{diag}(u)\mathbf{e}_i \\ &= \sum_{ij} u_i^2u_j^2\left(\mathbf{e}_i^\top \operatorname{Off}(XX^\top)\mathbf{e}_j\right)^2 \lesssim \|\Sigma\|^2\end{aligned}$$

Therefore, we conclude that

$$\mathsf{E}\left(u^\top \operatorname{Off}(YY^\top)u\right)^2 \lesssim \delta^2\|\Sigma\|^2.$$

As for the diagonal, we have

$$\mathsf{E}\left(u^\top \mathrm{Diag}(YY^\top)u\right)^2 = \mathsf{E}\left(\sum_i \delta^i u_i^2 X_i^2\right) = \delta^2 \mathsf{E}\left(u^\top \mathrm{Diag}(XX^\top)u\right)^2 + (\delta - \delta^2)\sum_i u_i^4 \mathsf{E}X_i^4$$
$$\lesssim \delta^2 \|\Sigma\|^2 + (\delta - \delta^2)\max_i \mathsf{E}X_i^4 \sum_i u_i^2 \lesssim \delta \|\Sigma\|^2.$$

□

Before we start with the proof of deviation bound let us present the following version of Talagrand's concentration inequality for the empirical processes, which will help us to capture the tail behavior in the subgaussian regime. Remarkably, the following result can be proven using very similar techniques: at first one may use the modified logarithmic Sobolev inequality to prove a version of Talagrand's concentration inequality in the bounded case and then use the truncation as in the proof of Theorem 4.1 to get the result in the unbounded case.

Theorem 4.4 (Theorem 4 in Adamczak (2008))**.** *Let $X_1, \ldots, X_N \in \mathscr{X}$ be independent sample and $\mathscr{F}$ is a countable class of measurable functions $\mathscr{X} \mapsto \mathbb{R}$ such that $\sup_{f \in \mathscr{F}} \|f(X_i)\|_{\psi_1} < \infty$. Set,*

$$Z = \sup_{f \in \mathscr{F}} \left| \sum_{i=1}^N f(X_i) - \mathsf{E} f(X_i) \right| \tag{4.43}$$

and $\sigma^2 = \sup_{f \in \mathscr{F}} \sum_{i=1}^N \mathsf{E} f^2(X_i)$. Then, there is an absolute constant $C > 0$ such that

$$\mathsf{P}(Z > 2\mathsf{E}Z + t) \le \exp\left(-\frac{t^2}{4\sigma^2}\right) + 3\exp\left(-\frac{t}{C\|\max_i \sup_f |f(X_i)|\|_{\psi_1}}\right).$$

Proof of Theorem 4.3. At first, using (4.37) we have

$$\|\hat{\Sigma} - \Sigma\| \lesssim \delta^{-1}\left\|\mathrm{Diag}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\,\mathrm{Diag}(\hat{\Sigma}^{(\delta)})\right\| + \delta^{-2}\left\|\mathrm{Off}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\,\mathrm{Off}(\hat{\Sigma}^{(\delta)})\right\|,$$

Let us apply our version of matrix Bernstein inequality to $N\,\mathrm{Off}(\hat{\Sigma}^{(\delta)}) = \sum_{i=1}^N \mathrm{Off}(Y_i Y_i^\top)$ with

$$R = CN\delta^2 \operatorname{tr}(\Sigma)(\Sigma + \mathrm{Diag}(\Sigma)).$$

We have $\tilde{\mathbf{r}}(R) \le 2\tilde{\mathbf{r}}(\Sigma)$ and $\|R\| \lesssim N\delta^2 \operatorname{tr}(\Sigma)\|\Sigma\|$. Therefore, with probability at least $1-e^{-t}$

$$\begin{aligned}\|\operatorname{Off}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Off}(\hat{\Sigma}^{(\delta)})\| &\lesssim \max\left(\sqrt{\frac{\delta^2 \operatorname{tr}(\Sigma)\|\Sigma\|(\log \mathtt{r}(\Sigma)+t)}{N}}, \frac{\operatorname{tr}(\Sigma)(\log \mathtt{r}(\Sigma)+t)\log N}{N}\right) \\ &= \|\Sigma\| \max\left(\sqrt{\frac{\delta^2 \mathtt{r}(\Sigma)(\log \mathtt{r}(\Sigma)+t)}{N}}, \frac{\mathtt{r}(\Sigma)(\log \mathtt{r}(\Sigma)+t)\log N}{N}\right). \end{aligned} \tag{4.44}$$

Integrating this bound (see e.g. Theorem 2.3 in Boucheron et al. (2013)) we easily get

$$\mathsf{E}\|\operatorname{Off}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Off}(\hat{\Sigma}^{(\delta)})\| \lesssim \|\Sigma\| \max\left(\sqrt{\frac{\delta^2 \mathtt{r}(\Sigma)\log \mathtt{r}(\Sigma)}{N}}, \frac{\mathtt{r}(\Sigma)\log \mathtt{r}(\Sigma)\log N}{N}\right).$$

Now we apply Theorem 4.4 to the set of functions indexed by $\gamma \in S^{n-1}$,

$$f_\gamma(X_i) = \gamma^\top \operatorname{Off}(Y_i Y_i^\top)\gamma,$$

so that $Z = N\|\operatorname{Off}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Off}(\hat{\Sigma}^{(\delta)})\|$ in (4.43). Then, by Lemma 4.10 we have $\sigma^2 \lesssim \delta^2 N\|\Sigma\|^2$ and by Lemma 4.9 $\|\max_i \sup_f |f(X_i)|\|_{\psi_1} = \|\max_i \|\operatorname{Off}(Y_i Y_i^\top)\|\|_{\psi_1} \lesssim \tilde{\mathbf{r}}(\Sigma)\|\Sigma\|\log N$, so that with probability $1-e^{-t}$ for $t \ge 1$

$$\begin{aligned}\|\operatorname{Off}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Off}(\hat{\Sigma}^{(\delta)})\| &\le 2\mathsf{E}\|\operatorname{Off}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Off}(\hat{\Sigma}^{(\delta)})\| + \delta\|\Sigma\|\sqrt{\frac{t}{N}} + \|\Sigma\|\frac{\tilde{\mathbf{r}}(\Sigma)t\log N}{N} \\ &\lesssim \|\Sigma\| \max\left(\sqrt{\frac{\delta^2 \mathtt{r}(\Sigma)\log \mathtt{r}(\Sigma)}{N}}, \sqrt{\frac{\delta^2 t}{N}}, \frac{\mathtt{r}(\Sigma)(\log \mathtt{r}(\Sigma)+t)\log N}{N}\right).\end{aligned}$$

We proceed with the diagonal term. Applying Proposition 4.3 to the sum $N\operatorname{Diag}(\hat{\Sigma}^{(\delta)}) = \sum_{i=1}^N \operatorname{Diag}(Y_i Y_i^\top)$ with $R = CN\delta \operatorname{tr}(\Sigma)\operatorname{Diag}(\Sigma)$ we have $\mathtt{r}(R) \lesssim \mathtt{r}(\Sigma)$ and $\|R\| \lesssim N\delta \operatorname{tr}(\Sigma)\|\Sigma\|$. Thus, with probability at least $1-e^{-t}$ we get,

$$\|\operatorname{Diag}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Diag}(\hat{\Sigma}^{(\delta)})\| \lesssim \|\Sigma\| \max\left(\sqrt{\frac{\delta \mathtt{r}(\Sigma)(\log \mathtt{r}(\Sigma)+t)}{N}}, \frac{\mathtt{r}(\Sigma)(\log \mathtt{r}(\Sigma)+t)\log N}{N}\right). \tag{4.45}$$

Again, integrating this inequality we get a bound for the expectation,

$$\mathsf{E}\|\operatorname{Diag}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Diag}(\hat{\Sigma}^{(\delta)})\| \lesssim \|\Sigma\| \max\left(\sqrt{\frac{\delta \mathtt{r}(\Sigma)\log \mathtt{r}(\Sigma)}{N}}, \frac{\mathtt{r}(\Sigma)\log \mathtt{r}(\Sigma)\log N}{N}\right).$$

We have $\|u^\top \operatorname{Diag}(Y_iY_i^\top)u\|_{L_2}^2 \lesssim \delta\|\Sigma\|^2$ and $\|\max_i \|\operatorname{Off}(Y_iY_i^\top)\|\|_{\psi_1} \lesssim \tilde{\mathbf{r}}(\Sigma)\|\Sigma\|\log N$ by Lemma 4.10 and Lemma 4.9. By Theorem 4.4 we have with probability at least $1-e^{-t}$,

$$\begin{aligned}\|\operatorname{Diag}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Diag}(\hat{\Sigma}^{(\delta)})\| &\le 2\mathsf{E}\|\operatorname{Diag}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Diag}(\hat{\Sigma}^{(\delta)})\| + \|\Sigma\|\sqrt{\frac{\delta t}{N}} + \|\Sigma\|\frac{\tilde{\mathbf{r}}(\Sigma)t\log N}{N} \\ &\lesssim \|\Sigma\|\max\left(\sqrt{\frac{\delta\mathtt{r}(\Sigma)\log\mathtt{r}(\Sigma)}{N}}, \sqrt{\frac{\delta t}{N}}, \frac{\mathtt{r}(\Sigma)(\log\mathtt{r}(\Sigma)+t)\log N}{N}\right).\end{aligned}$$

It is left to combine the off-diagonal and diagonal bounds,

$$\|\hat{\Sigma}-\Sigma\| \le \delta^{-2}\|\operatorname{Off}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Off}(\hat{\Sigma}^{(\delta)})\| + \delta^{-1}\|\operatorname{Diag}(\hat{\Sigma}^{(\delta)}) - \mathsf{E}\operatorname{Diag}(\hat{\Sigma}^{(\delta)})\|.$$

□

4.4 Approximation argument for non-smooth functions

In this section we explain how one can apply the Sobolev inequality for functions that are not everywhere diffirentiable rigorously. In order to use the Assumption (4.6), we need to take smooth approximations of the function

$$Z(X) = \sup_A (X^\top AX - \mathsf{E}X^\top AX).$$

Notice, that we have

$$|Z(X)-Z(Y)| \le \|X-Y\|\left(\sup_A \|AX\| + \sup_A \|AY\|\right).$$

The following simple lemma shows how to apply the logarithmic Sobolev inequality to non-differentiable functions that satisfy such inequality.

Lemma 4.11. *Suppose, a random vector X satisfies Assumption 4.1. Let $f:\mathbb{R}^n\to\mathbb{R}$ be such that*

$$|f(x)-f(y)| \le |x-y|\max(L(x),L(y)),$$

for some continuous $L(x)\ge 0$. Then, for some absolute constant $C>0$ and any $\lambda\in\mathbb{R}$ it holds,

$$\operatorname{Ent}(e^{\lambda f}) \le CK^2\lambda^2\mathsf{E}L(x)^2e^{\lambda f}$$

Proof. Set $h(x) = x^2(1-x)_+^2$ and consider a smoothing kernel supported on unit ball,

$$\phi(u) = \frac{1}{Z_h} h(\|u\|^2), \qquad Z_h = \int h(\|u\|^2) du = S_{n-1} \int_0^\infty h(r^2) dr,$$

where S_{n-1} is a surface area of the unit sphere in $\mathbb{R}^n$. Note, that since ϕ is radial, $\nabla\phi(u) = -\nabla\phi(-u)$ and also,

$$\int \|u\| \|\nabla\phi(u)\| du = \frac{2S_{n-1}}{Z_h} \int_0^\infty r^2 |g'(r)| dr = \frac{2\int_0^\infty r^2 |h'(r)| dr}{\int_0^\infty h(r^2) dr} = C_h.$$

Setting $\phi_m(u) = m^{-1}\phi(u/m)$ we have $\nabla\phi_m(u) = m^{-2}(\nabla\phi)(u/m)$, therefore

$$\int \|u\| \|\nabla\phi_m(u)\| du = \int \left\| \frac{u}{m} \right\| \left\| (\nabla\phi)\left(\frac{u}{m}\right) \right\| d\frac{u}{m} = C_h.$$

Take $F(x) = e^{\lambda f(x)/2}$ and let us consider a sequence of smooth approximations $F_m(x) = \int \phi_m(x-u) F(u) du$, so that $F_m(x)$ tends to F pointwise due to the fact that F is continuous. Moreover, we have due to the symmetry

$$\begin{aligned} \nabla F_m(x) &= \int (\nabla\phi_m)(x-u) F(u) du = \int (\nabla\phi_m)(u) F(x-u) du \\ &= \frac{1}{2} \int (\nabla\phi_m)(u) [F(x-u) - F(x+u)] du. \end{aligned}$$

Since $\phi_m(u)$ vanishes for $\|u\| \geq 1/m$, we have

$$\begin{aligned} \|\nabla F_m(x)\| &\leq \frac{1}{2} \sup_{\|u\| \leq m^{-1}} \frac{|F(x-u) - F(x+u)|}{\|u\|} \int \|u\| \|\nabla\phi_m(u)\| du \\ &\leq C_g \sup_{\|u\| \leq m^{-1}} \frac{|F(x-u) - F(x+u)|}{2\|u\|}. \end{aligned}$$

It is easy to see that

$$|F(x) - F(y)| = |e^{\lambda f(x)/2} - e^{\lambda f(y)/2}| \leq \|x-y\| \max(e^{\lambda f(x)/2}, e^{\lambda f(y)/2}) \max(L(x), L(y)),$$

therefore

$$\|\nabla F_m(x)\| \leq C_g \widetilde{F}_m(x) \times L_m(x),$$

where we set $L_m(x) = \sup_{y\,:\,\|x-y\| \leq m^{-1}} L(y)$ and $\widetilde{F}_m(x) = \sup_{\|x-y\| \leq m^{-1}} e^{\lambda f(y)/2}$, tend pointwise to $L(x)$ and $F(x)$, respectively, as $m \to \infty$. Since each f_m is smooth, we have by the

Assumption 1,

$$\mathrm{Ent}(F_m^2) \leq K^2 \mathsf{E}\|\nabla F_m(x)\|^2 \leq 2C_g K^2 \mathsf{E} L_m^2(x) \widetilde{F}_m(x)^2,$$

and taking limit $m \to \infty$ gives the required inequality. $\square$

Appendix A

Technical tools

A.1 Lasso and missing observations

Suppose, we observe a signal $\mathbf{y} \in \mathbb{R}^n$ of the form

$$\mathbf{y} = \Phi \mathbf{b}^* + \varepsilon,$$

where $\Phi = [\phi_1, \ldots, \phi_p] \in \mathbb{R}^{n \times p}$ is a dictionary of words $\phi_j \in \mathbb{R}^n$ and $\mathbf{b}^*$ is some sparse parameter with a support $\Lambda \subset \{1, \ldots, p\}$. We want to recover exact sparse representation by solving quadratic program

$$\frac{1}{2}\|\mathbf{y} - \Phi \mathbf{b}\|^2 + \gamma \|\mathbf{b}\|_1 \to \min_{\mathbf{b} \in \mathbb{R}^p}. \tag{A.1}$$

Denote by $\mathbb{R}^\Lambda$ the set of vectors with elements indexed by Λ, for $\mathbf{b} \in \mathbb{R}^n$ let $\mathbf{x}_\Lambda \in \mathbb{R}^\Lambda$ be the result of taking only elements indexed by Λ. With some abuse of notation we will also associate each vector $\mathbf{x}_\Lambda \in \mathbb{R}^\Lambda$ with a vector $\mathbf{x}$ from $\mathbb{R}^n$ that has same coefficients on Λ and zeros elsewhere. Let us also $\Phi_\Lambda = [\phi_j]_{j \in \Lambda}$ be a subdictionary composed of words indexed by Λ and P_Λ is the projector onto the corresponding subspace.

The following sufficient conditions for the global minimizer of (A.1) to be supported on Λ are due to Tropp (2006), who uses the notion of *exact recovery coefficient*,

$$\mathrm{ERC}_\Phi(\Lambda) = 1 - \max_{j \notin \Lambda} \|\Phi_\Lambda^+ \phi_j\|_1,$$

The results are summarized in the next theorem.

Theorem A.1 (Tropp (2006))**.** *Let* $\tilde{\mathbf{b}}$ *be a solution to* (A.1). *Suppose, that* $\|\Phi^\top \varepsilon\|_\infty \le \gamma \mathrm{ERC}(\Lambda)$. *Then,*

- *the support of* $\tilde{\mathbf{b}}$ *is contained in* Λ*;*
- *the distance between* $\tilde{\mathbf{b}}$ *and optimal (non-penalized) parameter satisfies,*

$$\|\tilde{\mathbf{b}} - \mathbf{b}^*\|_\infty \le \|\Phi_\Lambda^+ \varepsilon\|_\infty + \gamma \|(\Phi_\Lambda \Phi_\Lambda^\top)^{-1}\|_{1,\infty},$$
$$\|\Phi_\Lambda(\tilde{\mathbf{b}} - \mathbf{b}^*) - P_\Lambda \varepsilon\|_2 \le \gamma \|(\Phi_\Lambda^+)^\top\|_{2,\infty};$$

In what follows we want to extend this result for the possibility of using missing observations model. Observe that the program (A.1) is equivalent to

$$\frac{1}{2}\mathbf{b}^\top [\Phi^\top \Phi]\mathbf{b} - \mathbf{b}^\top [\Phi^\top \mathbf{y}] + \gamma \|\mathbf{b}\|_1 \to \min_{\mathbf{b}\in\mathbb{R}^p},$$

so that for the minimization procedure only knowing $D = \Phi^\top \Phi$ and $\mathbf{c} = \Phi^\top \mathbf{y}$ is required. Suppose, that instead we have only access to some estimators $\hat{D} \ge 0$ and $\hat{\mathbf{c}}$ that are close enough to the original matrix and vector, which may come e.g. from missing observations model. Then, we can solve instead the following problem,

$$\frac{1}{2}\mathbf{b}^\top \hat{D}\mathbf{b} - \mathbf{b}^\top \hat{\mathbf{c}} + \gamma \|\mathbf{b}\|_1 \to \min_{\mathbf{b}\in\mathbb{R}^p}. \tag{A.2}$$

In what follows we provide a slight extension of Tropp's result towards missing observations, the proof mainly follows the same steps.

Further, for a matrix D and two sets of indices A, B we denote the submatrix on those indices as $D_{A,B}$ and for a vector $\mathbf{c}$, the corresponding subvector is $\mathbf{c}_A$.

Lemma A.1. *Suppose, that*

$$\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\Lambda - \hat{\mathbf{c}}_{\Lambda^c}\|_\infty \le \gamma(1 - \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty}).$$

Then, the solution $\tilde{\mathbf{b}}$ *to* (A.2) *is supported on* Λ.

Proof. Let $\tilde{\mathbf{b}}$ be the solution to (A.2) with the restriction $\mathrm{supp}(\mathbf{b}) \subset \Lambda$. Since $\hat{D} \ge 0$ this is a convex problem and therefore the solution is unique and satisfy,

$$\hat{D}_{\Lambda,\Lambda}\tilde{\mathbf{b}} - \hat{\mathbf{c}}_\Lambda + \gamma \mathbf{g} = 0, \qquad \mathbf{g} \in \partial \|\tilde{\mathbf{b}}\|_1,$$

where $\partial f(\mathbf{b})$ denotes subdifferential of a convex function f at a point $\mathbf{b}$, in the case of ℓ_1 norm we have $\|\mathbf{g}\|_\infty \le 1$. Thus,

$$\tilde{\mathbf{b}} = \hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \gamma \hat{D}_{\Lambda,\Lambda}^{-1}\mathbf{g}. \tag{A.3}$$

Next, we want to check that $\tilde{\mathbf{b}}$ is a global minimizer. To do so, let us compare the objective function at a point $\overline{\mathbf{b}} = \tilde{\mathbf{b}} + \delta \mathbf{e}_j$ for arbitrary index $j \notin \Lambda$. Since $\|\overline{\mathbf{b}}\|_1 = \|\tilde{\mathbf{b}}\|_1 + |\delta|$, we have

$$\begin{aligned}
L(\tilde{\mathbf{b}}) - L(\overline{\mathbf{b}}) &= \frac{1}{2}\tilde{\mathbf{b}}^\top \hat{D}\tilde{\mathbf{b}} - \frac{1}{2}\overline{\mathbf{b}}^\top \hat{D}\overline{\mathbf{b}} - \hat{\mathbf{c}}^\top(\tilde{\mathbf{b}} - \overline{\mathbf{b}}) - \gamma|\delta| \\
&= \frac{\delta^2}{2}\mathbf{e}_j^\top \hat{D}\mathbf{e}_j + |\delta|\gamma - \delta \mathbf{e}_j^\top \hat{D}\tilde{\mathbf{b}} + \delta \widehat{c}_j \\
&> |\delta|\gamma - \delta \mathbf{e}_j^\top \hat{D}\tilde{\mathbf{b}} + \delta \widehat{c}_j,
\end{aligned}$$

where the latter comes from the fact that $\hat{D}$ is positively definite. Applying the equality (A.3) yields,

$$\mathbf{e}_j^\top \hat{D}\tilde{\mathbf{b}} = \hat{D}_{j,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \gamma \hat{D}_{j,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\mathbf{g},$$

therefore, taking into account $\|\mathbf{g}\|_\infty \le 1$ we have,

$$L(\tilde{\mathbf{b}}) - L(\overline{\mathbf{b}}) > |\delta| \left[\gamma(1 - \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty}) - \left|\hat{D}_{j,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \widehat{c}_j\right|\right],$$

where the right-hand side is nonnegative by the condition of the lemma. Since $j \notin \Lambda$ is arbitrary, $\tilde{\mathbf{b}}$ is a global solution as well.

□

Remark A.1. *It is not hard to see that in the exact case $\hat{D} = \Phi^\top \Phi$ and $\hat{\mathbf{c}} = \Phi^\top \mathbf{y}$ the condition of the lemma above turns into the condition $\|\Phi_{\Lambda^c}^\top P_\Lambda \varepsilon\|_\infty \le \gamma \mathrm{ERC}(\Lambda)$ of Theorem A.1.*

Since we are particularly interested in an application to time series, the features matrix Φ should in fact be random, thus stating a ERC-like condition onto it might result in additional unnecessary technical difficulties. Instead, let us assume that there is some other matrix $\bar{D}$, potentially the expectation of $\Phi^\top \Phi$, such that it is close enough to $\hat{D}$ (with some probability, but we are stating all the results deterministically in this section), and the value that controls the exact recovery looks like

$$\mathrm{ERC}(\Lambda; \bar{D}) = 1 - \|\bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty}.$$

Additionally, we set $\bar{\mathbf{c}} = \bar{D}\mathbf{b}^* = \bar{D}_{\cdot,\Lambda}\mathbf{b}^*_\Lambda$ — the vector that $\hat{\mathbf{c}}$ is intended to approximate. Note that in this case we have $\bar{D}_{\Lambda^c,\Lambda}\bar{D}^{-1}_{\Lambda,\Lambda}\bar{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_{\Lambda^c} = \bar{D}_{\Lambda^c,\Lambda}\mathbf{b}^*_\Lambda - \bar{\mathbf{c}}_{\Lambda^c} = 0$, thus the conditions of Lemma A.1 hold for $\bar{D}, \bar{\mathbf{c}}$ once $\mathrm{ERC}(\Lambda;\bar{D})$ and γ are nonnegative. In what follows we control the values appearing in the lemma for $\hat{D}$ and $\hat{\mathbf{c}}$ through the differences between $\bar{\mathbf{c}}$, $\bar{D}$ and $\hat{\mathbf{c}}$, $\hat{D}$, respectively, thus allowing the exact recovery of the sparsity pattern. Lemma 3.7

Corollary A.1. *Let* $\bar{D}$ *and* $\bar{\mathbf{c}}$ *be such that* $\bar{\mathbf{c}} = \bar{D}\mathbf{b}^*$. *Assume that*

$$\|\hat{\mathbf{c}} - \bar{\mathbf{c}}\|_\infty \le \delta_c, \qquad \|\bar{D}^{-1}_{\Lambda,\Lambda}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty \le \delta'_c, \qquad \|\bar{D}^{-1}_{\Lambda,\Lambda}(\hat{D}_{\Lambda,\cdot} - \bar{D}_{\Lambda,\cdot})\|_{\infty,\infty} \le \delta_D,$$
$$\|(\hat{D}_{\cdot,\Lambda} - \bar{D}_{\cdot,\Lambda})\mathbf{b}^*_\Lambda\|_\infty \le \delta'_D, \qquad \|\bar{D}^{-1}_{\Lambda,\Lambda}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}^*_\Lambda\|_\infty \le \delta''_D.$$

Suppose, $\mathrm{ERC}(\Lambda) \ge 3/4$ *and*

$$3\delta_c + 3\delta'_D \le \gamma, \qquad s\delta_D \le \frac{1}{16},$$

where $|\Lambda| = s$. *Then, the solution to* (A.2) *is supported on* Λ *and satisfies*

$$\tilde{\mathbf{b}}_\Lambda = \hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\Lambda - \gamma\hat{D}^{-1}_{\Lambda,\Lambda}\mathbf{g}, \tag{A.4}$$

with some $\mathbf{g} \in \mathbb{R}^s$ *satisfying* $\|\mathbf{g}_\Lambda\|_\infty \le 1$ *and the max-norm error satisfies*

$$\|\tilde{\mathbf{b}} - \mathbf{b}^*\|_\infty \le 2(\delta''_D + \delta'_c + \gamma\|\bar{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty}),$$

while the ℓ_2*-norm error satisfies*

$$\|\tilde{\mathbf{b}} - \mathbf{b}^*\| \le 2\sqrt{s}(\delta''_D + \delta'_c + \gamma\sigma^{-1}_{\min}).$$

If additionally $2(\delta''_D + \delta'_c + \gamma\|\bar{D}^{-1}_{\Lambda,\Lambda}\|_{1,\infty}) \le \min_{j\in\Lambda}|\mathbf{b}^*_j|$, *then we have the exact recovery, so that the following equality takes place*

$$\tilde{\mathbf{b}}_\Lambda = \hat{D}^{-1}_{\Lambda,\Lambda}\hat{\mathbf{c}}_\lambda - \gamma\hat{D}^{-1}_{\Lambda,\Lambda}\mathbf{s}_\Lambda,$$

where $\mathbf{s} = \mathrm{sign}(\mathbf{b}^*)$.

Proof. First observe that $D_{\Lambda^c,\Lambda}D_{\Lambda,\Lambda}^{-1}\mathbf{c}_\Lambda - \mathbf{c}_{\Lambda^c} = \Phi_{\Lambda^c}^\top(\Phi_\Lambda^+\mathbf{y}-\mathbf{y}) = \Phi_{\Lambda^c}^\top(P_\Lambda - I)\varepsilon$. By Lemma A.2 we have,

$$\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty} \le \|\bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty} + 4s\delta_D \le 1/2,$$

while since $\bar{\mathbf{c}}_{\Lambda^c} = \bar{D}_{\Lambda^c,\Lambda}\mathbf{b}_\Lambda^* = \bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda$,

$$\begin{aligned}
\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \hat{\mathbf{c}}_{\Lambda^c}\|_\infty &\le \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \bar{D}_{\Lambda^c,\Lambda}\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda\|_\infty + \|\hat{\mathbf{c}}_{\Lambda^c} - \bar{\mathbf{c}}_{\Lambda^c}\|_\infty \\
&\le \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty + \|\hat{D}_{\Lambda^c,\Lambda}(\hat{D}_{\Lambda,\Lambda}^{-1} - \bar{D}_{\Lambda,\Lambda}^{-1})\bar{\mathbf{c}}_\Lambda\|_\infty \\
&\quad + \|(\hat{D}_{\Lambda^c,\Lambda} - \bar{D}_{\Lambda^c,\Lambda})\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda\|_\infty + \delta_c \\
&\le \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty + \|\hat{D}_{\Lambda^c,\Lambda}(\hat{D}_{\Lambda,\Lambda}^{-1} - \bar{D}_{\Lambda,\Lambda}^{-1})\bar{\mathbf{c}}_\Lambda\|_\infty + \delta_D' + \delta_c.
\end{aligned}$$

Here, $\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty \le \delta_c/2$ due to $\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty} \le 1/2$. Moreover, we have

$$\begin{aligned}
\|\hat{D}_{\Lambda^c,\Lambda}(\hat{D}_{\Lambda,\Lambda}^{-1} - \bar{D}_{\Lambda,\Lambda}^{-1})\bar{\mathbf{c}}_\Lambda\|_\infty &= \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda\|_\infty \\
&\le \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty}\|(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda\|_\infty \\
&\le \delta_D'/2.
\end{aligned}$$

Using the condition on γ, we get that

$$\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \hat{\mathbf{c}}_{\Lambda^c}\|_\infty \le \frac{3}{2}(\delta_D' + \delta_c) \le \frac{\gamma}{2} \le \gamma(1 - \|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty}),$$

so that the conditions of Lemma A.1 are satisfied and (A.4) takes place. This allows us to write

$$\begin{aligned}
\tilde{\mathbf{b}}_\Lambda - \mathbf{b}_\Lambda^* &= \hat{D}_{\Lambda,\Lambda}^{-1}\hat{\mathbf{c}}_\Lambda - \bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda - \gamma\hat{D}_{\Lambda,\Lambda}^{-1}\mathbf{g}, \\
&= \hat{D}_{\Lambda,\Lambda}^{-1}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\bar{D}_{\Lambda,\Lambda}^{-1}\bar{\mathbf{c}}_\Lambda + \hat{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda) - \gamma\hat{D}_{\Lambda,\Lambda}^{-1}\mathbf{g} \\
&= \hat{D}_{\Lambda,\Lambda}^{-1}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}_\Lambda^* + \hat{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda) - \gamma\hat{D}_{\Lambda,\Lambda}^{-1}\mathbf{g} \\
&= \hat{D}_{\Lambda,\Lambda}^{-1}\bar{D}_{\Lambda,\Lambda}\left(\bar{D}_{\Lambda,\Lambda}^{-1}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}_\Lambda^* + \bar{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda) - \gamma\bar{D}_{\Lambda,\Lambda}^{-1}\mathbf{g}\right)
\end{aligned}$$

By Lemma A.2 we have $\|\hat{D}_{\Lambda,\Lambda}^{-1}\bar{D}_{\Lambda,\Lambda}\|_{\infty\mapsto\infty} \le 2$ so that

$$\|\tilde{\mathbf{b}}_\Lambda - \mathbf{b}_\Lambda^*\|_\infty \le 2\|\bar{D}_{\Lambda,\Lambda}^{-1}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}_\Lambda^*\|_\infty + 2\|\bar{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty + 2\gamma\|\bar{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty}.$$

and since we also have $|||\hat{D}_{\Lambda,\Lambda}^{-1}\bar{D}_{\Lambda,\Lambda}|||_{\mathsf{op}} \leq 2$ and $\|\mathbf{g}\| \leq \sqrt{s}$, it holds

$$\|\tilde{\mathbf{b}}_\Lambda - \mathbf{b}_\Lambda^*\| \leq 2\sqrt{s}\left(\|\bar{D}_{\Lambda,\Lambda}^{-1}(\bar{D}_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\mathbf{b}_\Lambda^*\|_\infty + \|\bar{D}_{\Lambda,\Lambda}^{-1}(\hat{\mathbf{c}}_\Lambda - \bar{\mathbf{c}}_\Lambda)\|_\infty + \gamma|||\bar{D}_{\Lambda,\Lambda}^{-1}|||_{\mathsf{op}}\right).$$

□

Before we proceed with the proof of this corollary, we present a technical lemma that collects some trivial inequalities.

Lemma A.2. *Set* $\delta_c = \|\hat{\mathbf{c}} - \bar{\mathbf{c}}\|_\infty$, $\delta_D = \|(\hat{D}_{\Lambda^c,\Lambda} - \bar{D}_{\Lambda^c,\Lambda})\bar{D}_{\Lambda,\Lambda}^{-1}\|_{\infty,\infty}$. *Suppose,* $\|\bar{D}_{\Lambda^c\Lambda}\bar{D}_{\Lambda\Lambda}^{-1}\|_{1,\infty} \leq 1$ *and* $s\delta_D \leq 1/2$. *It holds,*

- *for each* $q \geq 1$

$$\|D_{\Lambda,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{q\to q} \leq 2, \qquad \|\hat{D}_{\Lambda,\Lambda}^{-1}D_{\Lambda,\Lambda}\|_{q\to q} \leq 2\,;$$

- $$\|\hat{D}_{\Lambda^c,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1} - D_{\Lambda^c,\Lambda}D_{\Lambda,\Lambda}^{-1}\|_{1,\infty} \leq 4s\delta_D.$$

Proof. First, we have

$$\begin{aligned}
\|D_{\Lambda,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{q\to q} &= \|I + (D_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})\hat{D}_{\Lambda,\Lambda}^{-1}\|_{q\to q} \\
&\leq 1 + \|(D_{\Lambda,\Lambda} - \hat{D}_{\Lambda,\Lambda})D_{\Lambda,\Lambda}^{-1}\|_{q\to q}\|D_{\Lambda,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{q\to q} \\
&\leq 1 + s\delta_D\|D_{\Lambda,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{q\to q},
\end{aligned}$$

which solving the inequality and since $s\delta_D \leq 1/2$ turns into

$$\|D_{\Lambda,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{q\to q} \leq \frac{1}{1 - s\delta_D} \leq 2.$$

Similarly, $\|\hat{D}_{\Lambda,\Lambda}^{-1}D_{\Lambda,\Lambda}\|_{q\to q} \leq 2$.

Furthermore,

$$\begin{aligned}
\|(\hat{D}_{\Lambda^c,\Lambda} - D_{\Lambda^c,\Lambda})\hat{D}_{\Lambda,\Lambda}^{-1}\|_{1,\infty} &\leq \|(\hat{D}_{\Lambda^c,\Lambda} - D_{\Lambda^c,\Lambda})D_{\Lambda,\Lambda}^{-1}\|_{1,\infty}\|D_{\Lambda,\Lambda}\hat{D}_{\Lambda,\Lambda}^{-1}\|_{1\to 1} \\
&\leq 2s\delta_D.
\end{aligned}$$

and

$$\begin{aligned}\|D_{\Lambda^c,\Lambda}(D_{\Lambda,\Lambda}^{-1}-\hat{D}_{\Lambda,\Lambda}^{-1})\|_{1,\infty} \leq &\|D_{\Lambda,\Lambda^c}D_{\Lambda,\Lambda}^{-1}\|_{1,\infty}\|\hat{D}_{\Lambda,\Lambda}^{-1}(\hat{D}_{\Lambda,\Lambda}-D_{\Lambda,\Lambda})\|_{1\to1}\\ \leq &\|D_{\Lambda,\Lambda^c}D_{\Lambda,\Lambda}^{-1}\|_{1,\infty}\|\hat{D}_{\Lambda,\Lambda}^{-1}D_{\Lambda,\Lambda}\|_{1\to1}\|D_{\Lambda,\Lambda}^{-1}(\hat{D}-D)\|_{1\to1}\\ \leq &2\|D_{\Lambda,\Lambda^c}D_{\Lambda,\Lambda}^{-1}\|_{1,\infty}s\delta_D,\end{aligned}$$

which together give us the second inequality. □

A.2 Gaussian approximation for change point statistic

Let $X_1,\dots,X_n \in \mathbb{R}^d$ be a martingale difference sequence (MDS) with coefficients b_k, and set

$$\begin{aligned}\overline{\sigma}^2(q) &= \max_{j=1,\dots,d}\max_I \operatorname{Var}\left(q^{-1/2}\sum_{i\in I}X_{ij}\right),\\ \underline{\sigma}^2(q) &= \min_{j=1,\dots,d}\min_I \operatorname{Var}\left(q^{-1/2}\sum_{i\in I}X_{ij}\right),\end{aligned}$$

where $\max_I, \min_I$ are taken with respect to the subsets $I \subset \{1,\dots,n\}$ of form $I=\{i+1,\dots,i+q\}$. Let additionally, with probability one

$$|X_{ij}| \leq D_n, \qquad 1\leq i\leq n; 1\leq j\leq p.$$

Denote the statistics,

$$\check{T} = \max_{j=1,\dots,d} n^{-1/2}\sum_{i=1}^n X_{ij}, \tag{A.5}$$

and let $\check{Y}=(\check{Y}_1,\dots,\check{Y}_d)^\top$ be normal with zero mean and covariance $\mathsf{E}\check{Y}\check{Y}^\top = \Sigma := \frac{1}{n}\sum_{i=1}^n \mathsf{E}X_iX_i^\top$.

Theorem A.2 (Chernozhukov et al. (2013), Theorem B.1)**.** *Suppose, positive r,q be such that $r+q\leq n/2$ and for some $c_1,C_1>0$ and $0<c_2<1/4$, $c_1\leq\underline{\sigma}(q)\leq\overline{\sigma}(q)\vee\overline{\sigma}(r)\leq C_1$ for each $i=1,\dots,n$, $j=1,\dots,d$, $(r/q)\log^2 d\leq C_1n^{-c_2}$ and,*

$$\max\left\{qD_n\log^{1/2}d, rD_n\log^{3/2}d, \sqrt{q}D_n\log^{7/2}d\right\}\leq C_1n^{1/2-c_2}.$$

Then, there are $c,C>0$ that only exist on c_1,c_2,C_1, such that

$$\sup_t\left|\mathsf{P}(\check{T}<t)-\mathsf{P}(\max_{j\leq d}\check{Y}_j<t)\right|\leq Cn^{-c}+2(n/q-1)b_r.$$

Suppose we have another MDS $X'_1,\dots,X'_n$, from which we construct a similar to (A.5) statistic $\check{T}'$. Suppose, the sequence has β-mixing coefficients bounded by the same values b_k and the values of the vectors bounded a.s. by the same D_n. Finally, let us set $\Sigma' = \frac{1}{n}\sum_{i=1}^{n}\mathsf{E}X_iX_i^\top$. Combining the result above with Gaussian comparison and anti-concentration we get the following corollary.

Lemma A.3. *Suppose, there are positive q,r such that $q+r<n/2$ and there are $c_1,C_1>0$ and $0<c_2<1/4$ such that $c_1 \le \underline{\sigma}(q) \le \overline{\sigma}(q) \vee \overline{\sigma}(r) \le C_1$ holds for both (X_i), (X'_i). Let $|\Sigma_{jk}-\Sigma'_{jk}| \le \Delta$ for each $j,k=1,\dots,d$. Then, under conditions of Theorem A.2 it holds for each $t,\delta\in\mathbb{R}$,*

$$\left|\mathsf{P}(\check{T} > t+\delta) - \mathsf{P}(\check{T}' > t)\right| \le C\Delta^{1/3}\log^{2/3}p + C|\delta|\log^{1/2}p + Cn^{-c} + 2(n/q-1)b_r,$$

where $c,C>0$ only depend on c_1,c_2,C_1.

Proof. Simply apply Theorem A.2, together with Theorem 2 of Chernozhukov et al. (2015) and Theorem 1 of Chernozhukov et al. (2017). $\square$

Let now $X_1,\dots,X_n\in\mathbb{R}^p$ be a martingale difference sequence, with β-mixing coefficients b_k and $\mathrm{Var}(X_i)=V$. We need to bring the statistics

$$\hat{T} = \max_{s\in\mathscr{S}} \frac{1}{\sqrt{n}}\left\|\sqrt{\frac{n-s}{s}}\sum_{i=1}^{s}X_i - \sqrt{\frac{s}{n-s}}\sum_{i=s+1}^{n}X_i\right\|$$

into the above form. Following Zhilova (2015) we consider the following approximation. Let G_ε be an ε-net of the unit sphere in $\mathbb{R}^p$, such that for each $\mathbf{a}\in\mathbb{R}^p$ it holds,

$$(1-\varepsilon)\|\mathbf{a}\| \le \max_{\gamma\in G_\varepsilon}\gamma^\top\mathbf{a} \le (1+\varepsilon)\|\mathbf{a}\|.$$

Let $G_\varepsilon = \{\gamma_1,\dots,\gamma_{|G_\varepsilon|}\}$ be fixed and set,

$$[X]_{G_\varepsilon} = (\gamma_1^\top X,\dots,\gamma_{|G_\varepsilon|}^\top X) \in \mathbb{R}^{|G_\varepsilon|},$$

and having $\mathscr{S} = \{s_1 < s_2 < \dots < s_{|\mathscr{S}|}\}$ set for each $i = 1, \dots, n$ a stacked vector,

$$\widetilde{X}_i = \left(\alpha_{n,s_1}(i)[X_i]_{G_\varepsilon}^\top, \dots, \alpha_{n,s_{|S|}}(i)[X_i]_{G_\varepsilon}^\top\right)^\top \in \mathbb{R}^{|S| \times |G_\varepsilon|},$$
$$\alpha_{n,s}(i) = sign(s - i + 1/2)\left(\frac{n-s}{s}\right)^{sign(s-i+1/2)/2},$$

which implies that

$$(1-\varepsilon)\hat{T} \le \max_j \frac{1}{\sqrt{n}} \sum_{i=1}^n \widetilde{X}_{ij} \le (1+\varepsilon)\hat{T}.$$

For sake of simplicity assume, $a^{-1} \le s/(n-s) \le a$ for each $s \in \mathscr{S}$. Note, that for each j and $|I| = q$ it holds for some γ that,

$$\operatorname{Var}\left(q^{-1/2} \sum_{i \in I} \widetilde{X}_{ij}\right) = \operatorname{Var}\left(q^{-1/2} \sum_{i \in I} \gamma^\top X_i\right) \in [\sigma_{\min}(V), \sigma_{\max}(V)].$$

Suppose, there is another MDS $X_1', \dots, X_n'$ with same mixing properties and set for each interval I of observations,

$$V_I' = \frac{1}{q} \sum_{i \in I} \mathsf{E} X_i' [X_i']^\top, \qquad |I| = q,$$

and assume that for each such I it holds,

$$\|V_I' - V\| \le \Delta_I, \qquad \Delta_q = \max_{|I|=q} \Delta_I.$$

Denote by analogy the test statistics $\hat{T}'$ and the vectors $\widetilde{X}_i'$. In what follows we assume that the dimension p is constant and the size of $\mathscr{S}$ is growing with n. Moreover, assume that $|X_{ij}|, |X_{ij}'| \le D_n$ for each i, j and that $\hat{T}, \hat{T}' \le A_n$, all with probability $\ge 1 - 1/n$.

Lemma A.4. *Suppose, positive r, q be such that $r + q \le n/2$ and for some $c_1, C_1 > 0$ and $0 < c_2 < 1/4$, $c_1 \le \sigma_{\min}(V) \le \sigma_{\max}(V) \le C_1$ for each $i = 1, \dots, n$, $j = 1, \dots, d$, $(r/q)\log^2 n \le C_1 n^{-c_2}$ and,*

$$\max\left\{qD_n \log^{1/2} n, rD_n \log^{3/2} n, \sqrt{q} D_n \log^{7/2} n\right\} \le C_1 n^{1/2 - c_2}.$$

Moreover, assume $\Delta_r, \Delta_q \leq c_1/2$. Then, for any $C_2 > 0$ there are $c, C > 0$ that only depend on c_1, c_2, C_1, C_2, such that for each $t, \delta \in \mathbb{R}$ it holds,

$$\begin{aligned}\left|\mathsf{P}(\hat{T} > t+\delta) - \mathsf{P}(\hat{T}' > t)\right| \leq & C\Delta^{1/3}\log^{2/3} n + C(A_n n^{-C_2} + |\delta|)\log^{1/2} n \\ & + Cn^{-c} + 2(n/q-1)b_r,\end{aligned}$$

where $\Delta = \max_{s\in\mathscr{S}}\{\Delta_{[1,s]}, \Delta_{(s,n]}, \Delta_n\}$.

Proof. Take $\varepsilon = n^{-C_2}$, then we can have $\log|G_\varepsilon| \lesssim \log n$, so that if d is dimension of $\widetilde{X}$, then $\log p \lesssim \log n$. In order to apply Lemma A.3 with $\delta = \varepsilon A_n + \delta$, it is left to bound the covariance difference Δ. We have, that (assuming $s_1 \leq s_2$)

$$\begin{aligned}\frac{1}{n}\sum_{i=1} n\mathsf{E}\widetilde{X}_{ij}\widetilde{X}_{ik} =& \frac{1}{n}\sum_{i=1}^{n} a_{s_1,n}(i)a_{s_2,n}(i)\gamma_1^\top \mathsf{E}X_iX_i^\top\gamma_2 \\ =& \gamma_1^\top\left[\frac{s_1\frac{n-s_1}{s_1}\frac{n-s_2}{s_2} - (s_2-s_1)\frac{s_1}{n-s_1}\frac{n-s_2}{s_2} + (n-s_2)\frac{s_1}{n-s_1}\frac{s_2}{n-s_2}V}{n}\right]\gamma_2,\end{aligned}$$

while

$$\begin{aligned}\frac{1}{n}\sum_{i=1} n\mathsf{E}\widetilde{X}'_{ij}\widetilde{X}'_{ik} =& \frac{1}{n}\sum_{i=1}^{n} sign(s_1-i+1/2)sign(s_2-i+1/2)\gamma_1^\top\mathsf{E}X'_i[X'_i]^\top\gamma_2 \\ =& \gamma_1^\top\left[\frac{s_1\frac{n-s_1}{s_1}\frac{n-s_2}{s_2}V_{[1,s_1]} - (s_2-s_1)\frac{s_1}{n-s_1}\frac{n-s_2}{s_2}V_{(s_1,s_2]}}{n}\right. \\ & \left. + \frac{(n-s_2)\frac{s_1}{n-s_1}\frac{s_2}{n-s_2}V_{(s_2,n]}}{n}\right]\gamma_2.\end{aligned}$$

Observe, that $(s_2-s_1)V_{(s_1,s_2]} = nV_{[1,n]} - s_1V_{[1,s_1]} - (n-s_2)V_{(s_2,n]}$. Therefore, the difference between two is bounded by,

$$\begin{aligned}|\Sigma_{jk} - \Sigma'_{jk}| \leq & \frac{a^2 s_1}{n}\|V_{[1,s_1]} - V\| + \frac{a^2(n-s_2)}{n}\|V_{(s_2,n]} - V\| + a^2\|V_{[1,n]} - V\| \\ \leq & 2a^2\max_{s\in\mathscr{S}}\{\Delta_{[1,s]}, \Delta_{(s,n]}, \Delta_n\},\end{aligned}$$

thus the statement follows. $\square$

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Declaration

I hereby declare that I completed this work without any improper help from a third party and without using any aids other than those cited. All ideas derived directly or indirectly from other sources are identified as such. The results of Chapter 2 are based on joint work with Wolfgang Härdle and Xiu Xu. The results of Chapter 3 are based on joint work with Wolfgang Härdle and Cathy Chen. Finally, the results of Chapter 4 are based on joint paper with Nikita Zhivotovsky.

I testify through my signature that all information that I have provided about resources used in the writing of my doctoral thesis, about the resources and support provided to me as well as in earlier assessments of my doctoral thesis correspond in every aspect to the truth.

Berlin, den September 17, 2019 Yegor Klochkov